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The cooperative endorsement of a strategic game

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Abstract

This note provides a way to translate a strategic game to a characteristic cooperative game assuming that the set of players of the cooperative game is the set of pure actions of the strategic game. Coalitions generated with only one action for each player and the total coalition characterize the Core. We calculate the worth of the total coalition to guarantee the non-emptyness condition. In particular, for a two-player game, this value is equal to the maximal sum of the diagonals.

Keywords: Cooperative games, Core.

JEL: C7

1 Introduction

Strategic and cooperative behavior are not mutually exclusive as numerous papers have shown (e.g. Nash 1953, Raiffa 1953, Selten 1960 Aumann 1961). There is also extensive evidence in the empirical front that, indeed, agents act in a cooperative way

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(e.g. Roth 1979, Rabin 1993, Binmore 1994, Fehr and Schmidt 1999, Camerer 2003). Consistent with these two streams of the literature, is the notion that some agents prefer cooperating or, in other words, they can be benevolent with other agents in strategic situations. In light of this, we ask ourselves how those benevolent agents would play in a strategic situation formalized as a normal form game if they were considering that either their payoff or that of their opponents could be transferred, (i.e., a cooperative game with transferable utility, TU-games).

This note presents a cooperative game approach to study strategic situations with agents of a marked cooperative profile. Namely, we inquire about the minimal value that highly cooperative agents would need to support agreements immune to blockades of coalitions. Given a normal form game we construct a new cooperative game such that i) the set of agents in the new cooperative game is the union of all pure actions of the original game; ii) the characteristic function is tailored such that any collection of actions gives the maximal worth. In particular, it is the maximal of the sum of the payoffs associated to this action profile. The first assumption follows from Harsanyi's interpretation of a bayesian player playing an incomplete game. The fictitious agents are defined following the type-agent representation of Bayesian games suggested by Harsanyi: the actions of each agent are interpreted as their type in such class of games. In our case, each action could be considered as a different type where the opponent has the information on the set of types for each player. The second assumption establishes a positive case. Using an utilitarian criterion, we extrapolate the behavior of the players. If agents are strongly cooperative, they see their payoff as the opponent payoff and viceversa. Therefore, the sum of their payoffs is the best combination of their achievement.

Our result establishes the existence of the total coalition value in order to guarantee the existence of the Core solution accepted in the literature. Moreover, we present a characterization of such a value using only the relevant coalition for the non-cooperative game. In other words, coalitions generated with only one action for each player and the total coalition matter. In particular, for a two-player game, this value is equal to the maximal sum of the diagonals.

In Cooperative Games, there is a wide literature which gives a strategic interpretation to cooperative solutions (see the survey of Highland, 2009). Our approach is the opposite. In a similar vein, Ui (2000) gives a relationship between strategic games with potential and the Shapley value of a particular class of cooperative games indexed by the set of strategy profiles. This fact is in contrast with the large literature that gives strategic interpretation to cooperative solutions (see the survey of Serrano, 2009).

The note is organized as follows. Section 2 presents the construction of a family of cooperative games from a strategic game. Section 3 characterizes the endorsement associated to a strategic game by characterizing the Core of the cooperative game.

Finally, the computation of such value for a two-player game closes the note.

2 From (G, u) to $\{\Gamma, v\}_\alpha$

Let $(G, u) = (N, S_i, u_i : \Pi_{i\in N} S_i \to \mathbb{R})$ be a strategic game where $N = \{1, \ldots, n\}$ is the finite set of players, $S_i = \{i_1, \ldots, i_m\}$ the action set and u_i represents the utility function of player i. From (G, u) we generate a family of coalitional games denoted by $\{\Gamma(\alpha)\}_{\alpha\in\mathbb{R}^+}$. Each coalitional game in such family is defined by both the set of players and the characteristic function v. The function v assigns a worth to any possible coalition of players. First, let's translate any action k of each player i denoted by i_k to an agent of the new coalitional cooperative game. We call these agents as types. Therefore, each action of player i from the original game G corresponds with a type at the new coalition game Γ. Denote by $N^n = \cup_{i\in N} S_i$ the set of agents in Γ and for all coalition $S \in N^n$, and $\rho_i(S) = S \cap S_i$ corresponds with the actions that player i participates.

Define $\Gamma(\alpha) = (N^n, v^\alpha)$ with $v^\alpha : 2^{N^n} \to \mathbb{R}$ where

$$\begin{aligned}
v^\alpha(S) &= 0 \text{ if } \exists i \in N \text{ such that } \rho_i(S) = \emptyset \quad [1] \\
v^\alpha(S) &= Max\left(\sum_i u_i(t_1, t_2, \ldots, t_N)\ ,\ t_i \in \rho_i(S)\right) \text{ if } \forall i \in N : \rho_i(S) \neq \emptyset \\
v^\alpha(N^n) &= \alpha
\end{aligned}$$

The first condition says that any coalition of types which does not represent a possible action profile in G has zero worth. The second one sets up the worth of a coalition S where any player is active by at least one action, i.e.: $\rho_i(S) \neq \emptyset$. In particular, if $|S| = n$ and $\rho_i(S) \neq \emptyset$, $\forall i \in N$ then $v^\alpha(S) = \sum_i u_i(\rho_1(S), \ldots, \rho_N(S))$. The last condition states the worth of the total coalition. We call α the endorsement of the game G and we look for conditions on α to guaratee cooperative solution in $\Gamma(\alpha)$.

3 The endorsement of cooperation

This section presents the characterization of the Core of $\Gamma(\alpha)$ denoted by $C(\Gamma(\alpha))$. In order to prove proposition 1, we apply the characterization of balanced games to state the existence of non-empty Core. Thereafter, we propose a way to describe the Core using coalitions in $\Gamma(\alpha)$ where each player participates with only one of her actions. Finally, we state the minimum α for 2-players game in order to guarantee a non-emptiness Core. We call this amount as the endorsement of G .

Proposition 1 *There exists* $\alpha > 0$ *such that* $C(\Gamma(\alpha)) \neq \emptyset$.

Proof. Notice that $\Gamma(\alpha)$ is a balanced game for an α large enough, namely, for $\alpha = \sum_{S \in 2^{N^n}} v(S)$. By using Bondavera-Shapley theorem (M. J. Osborne and A. Rubinstein,1995. ppp 262 Proposition 262.2), $\Gamma(\alpha)$ has nonempty core. ■

Given that family of cooperative games $\{\Gamma(\alpha)\}_\alpha$ with non empty Core, by continuity and zero bounded restriction, it is easy to check the existence of the solution of the problem $\min(\alpha)$ subject to $Core(\Gamma(\alpha)) \neq \emptyset$. Let $\tilde{\alpha}$ be the solution of the above problem.

The value $\tilde{\alpha}$ represents the minimal investment in order to preclude the blocking of coalitions, in particular those coalitions linked to a profile of action of G. Let $K = \{S \subset N^n,\ |S| = n \text{ and } S \cap S_i \neq \emptyset \text{ for all } i \in N\}$ be the set[1] of coalitions with non-zero worth. From K, we define a new set $K(\Gamma(\alpha))$ which consist of the set of imputations determined by inequalities written only for coalitios in K. Therefore, any profile of actions in G represents a inequality in $K(\Gamma(\alpha))$. Formally:

$$K(\Gamma(\alpha)) = \left\{ x \in R_+^{N^n} : \sum_{i \in S} x_i \geq v^\alpha(S)\ \forall S \in K \text{ and } \sum_{i \in N^n} x_i = \alpha \right\}$$

The proposition below states that the Core of $\Gamma(\alpha)$ is equal to the set $K(\Gamma(\alpha))$.

Proposition 2 $C(\Gamma(\alpha)) = K(\Gamma(\alpha))$

Proof. It is straightforward that $C(\Gamma(\alpha)) \subset K(\Gamma(\alpha))$. In order to prove that $K(\Gamma(\alpha)) \subset C(\Gamma(\alpha))$, it is enough to see that

$$\sum_{i \in S} x_i \geq v^\alpha(S) \text{ if } \rho_i(S) \neq \emptyset \text{ for all } i \text{ and } \quad |S| > n$$

for all imputation $x \in K(\Gamma(\alpha))$.

Let $S \in N^n$ such that $|S| = n + 1$ and $\rho_i(S) \neq \emptyset$ for all $i \in N$. Suppose w.l.o.g that the first action for all players is in S and the second action of player 1, i.e.: $S = \{1_1, 1_2, 2_1, 3_1, \ldots, n_1\}$. By definition of the characteristic function v^α:

$$v^\alpha(S) = Max(\sum_{i \in N} u_i(1_1, 2_1, 3_1, \ldots, n_1)\ , \sum_{i \in N} u_i(1_2, 2_1, 3_1, \ldots, n_1)\ )$$

[1]Notice that K is isomorphic to the cartesian product of S_i, i.e.: $K = \times_{i \in N} S_i$.

If $x \in K(\Gamma(\alpha))$ it satisfies that:

$$\begin{aligned} \sum_{i \in N} x_{i_1} &\geq u_i(1_1, 2_1, 3_1, \ldots, n_1) \\ x_{1_2} + \sum_{i \in N \setminus \{1\}} x_{i_1} &\geq u_i(1_2, 2_1, 3_1, \ldots, n_1) \end{aligned}$$

by the non negative condition, we get:

$$\begin{aligned} \sum_{i \in N} x_{i_1} + x_{1_2} &\geq u_i(1_1, 2_1, 3_1, \ldots, n_1) \\ x_{1_1} + x_{1_2} + \sum_{i \in N \setminus \{1\}} x_{i_1} &\geq u_i(1_2, 2_1, 3_1, \ldots, n_1) \end{aligned}$$

Therefore,

$$x_{1_1} + x_{1_2} + \sum_{i \in N \setminus \{1\}} x_{i_1} \geq Max\left(\sum_{i \in N} u_i(1_1, 2_1, 3_1, \ldots, n_1)\ , \sum_{i \in N} u_i(1_2, 2_1, 3_1, \ldots, n_1)\ \right) = v^{\alpha}(S)$$

Therefore $x \in C(\Gamma(\alpha))$ and the result hold. ■

The next proposition depicts the value $\tilde{\alpha}$ for the family of two players games. In particular, the value $\tilde{\alpha}$ is either the sum of the principal diagonal of the payoff matrix or the sum of the other diagonal.

Consider the following notation for the matrix payoff of $G = (\{1, 2\}, S_i = \{0, 1\}, u_i)$ a two person game:

	2_0	2_1
1_0	a_{00}, b_{00}	a_{01}, b_{01}
1_1	a_{10}, b_{10}	a_{11}, b_{11}

Proposition 3 *Suppose that* $a_{00} + b_{00} \geq a_{ij} + b_{ij}\ \forall\ (i, j) \neq (0, 0)$. *Then* $\tilde{\alpha} = a_{00} + b_{00} + a_{11} + b_{11}$ *or* $\tilde{\alpha} = a_{01} + b_{01} + a_{10} + b_{10}$.

Proof. Given that $\tilde{\alpha}$ is the lowest value of the total coalition such that $Core(\Gamma(\alpha))$ is not empty, let's solve the following problem:

$$\begin{aligned} Min \quad & x_{i_0} + x_{i_1} + x_{j_0} + x_{j_1} \\ s.t. \quad & x_{i_0} + x_{j_0} \geq a_{00} + b_{00} \\ & x_{i_0} + x_{j_1} \geq a_{01} + b_{01} \\ & x_{i_1} + x_{j_0} \geq a_{10} + b_{10} \\ & x_{i_1} + x_{j_1} \geq a_{11} + b_{11} \end{aligned}$$

The min contition allows us to write the problem as:

$$\begin{array}{lll} Min & a_{00}+b_{00}+x_{i_1}+x_{j_1} & \\ s.a. & x_{i_0}+x_{j_1}\geq & a_{01}+b_{01} \\ & x_{i_1}+x_{j_0}\geq & a_{10}+b_{10} \\ & x_{i_1}+x_{j_1}\geq & a_{11}+b_{11} \end{array}$$

Adding the first and the second restriction, we obtain that $x_{i_1}+x_{j_1}\geq a_{01}+b_{01}+a_{10}+b_{10}-a_{00}-b_{00}$. Given the first condition, then the minimum is:

$$\tilde{\alpha}=a_{00}+b_{00}+\max\{a_{11}+b_{11},a_{01}+b_{01}+a_{10}+b_{10}-a_{00}-b_{00}\}$$

and the result holds. ■

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