

Discounted Solidarity Values

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>Emilio Calvo

Department of Economic Analysis and ERI-CES. Universitat de
Valencia. Spain.

>Esther Gutiérrez-López

Departamento de Economía Aplicada IV. Universidad del País
Vasco UPV/EHU. Spain

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Emilio Calvo*

Esther Gutiérrez-López†

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Abstract

We consider the family of discounted solidarity values Sl^α , where $\alpha \in [0, 1]$. We offer strategic support for this family by means of a noncooperative bargaining game. We show that the risk of a breakdown in negotiations and the time discount factor simultaneously determine the value of α . We supplement the analysis with an axiomatic characterization.

Keywords: n -person bargaining; transferable utility games; Solidarity value.

JEL Classification: C71

1 Introduction

In this paper we introduce a new parametric family of values for cooperative games called *discounted solidarity values* Sl^α , where $\alpha \in [0, 1]$. The range of this family goes from the Solidarity value, when $\alpha = 1$, to the equal split value, $E(N, v) = \frac{v(N)}{n}$, when $\alpha = 0$. We offer strategic support for this family by means of a noncooperative bargaining game. We show that the risk of a breakdown in negotiations and the time discount factor simultaneously determine the value of α .

The results presented here are parallel to those obtained in Calvo and Gutiérrez-López (2016), where the family of *discounted Shapley values* introduced by Joosten (1996) is considered.

Given a cooperative game (N, v) , the α -discounted Shapley value Sh^α of a coalitional game (N, v) is the Shapley value Sh (Shapley, 1953) of the coalitional game (N, v_α) , such that for any $S \subseteq N$, $v_\alpha(S) = \alpha^{n-s}v(S)$. That is, $Sh^\alpha(N, v) = Sh(N, v_\alpha)$.

In van den Brink and Funaki (2010) a bargaining game that implements the family of discounted Shapley values Sh^α is offered. There, the bidding mechanism given by Pérez-Castrillo and Wettstein (2001), which implements the Shapley value, is considered. Van den Brink and Funaki introduce a parameter $\alpha \in [0, 1]$ as a discount factor that determines the discounting of the available worth going from one round of negotiation to the next after a rejection of a proposal, i.e. if the proposal of player i to share $v(N)$ is rejected then the remaining agents $j \in N \setminus i$ continue the process, bargaining over $\alpha v(N \setminus i)$. Calvo and Gutiérrez-López (2016) offers alternative strategic support for Sh^α . We consider the bargaining game introduced by Hart and Mas-Colell (1996). This is an alternating random-proposer

*Department of Economic Analysis and ERI-CES. Universitat de Valencia, Spain. E-mail: emilio.calvo@uv.es.

†Departamento de Economía Aplicada IV. Universidad del País Vasco U.P.V./E.H.U., Spain. E-mail: mari-aester.gutierrez@ehu.es

bargaining model with a risk of breakdown $(1 - \rho)$. There, a proposer is chosen randomly. He makes an offer and the remaining players accept or reject the proposal. If all the remaining players accept the proposal, the game ends. If any respondent rejects the offer, then with a probability ρ , another proposer is selected randomly, and with probability $(1 - \rho)$, breakdown occurs: the proposer leaves the game with a payoff of zero and the remaining players restart the bargaining process. This is a sequential, perfect information game, and it has a stationary subgame perfect equilibrium. The *average* of the equilibrium proposals is the Shapley value Sh ; and when the probability of continuing the game ρ goes to one, the *limit* of these equilibrium proposals is also the Shapley value. We show that if all players have the same discount factor of $\delta \in [0, 1]$ ¹, when the risk of breakdown, $(1 - \rho)$, and the discount factor, δ , are considered simultaneously the subgame perfect equilibrium of the bargaining implements the discounted Shapley value Sh^α , where

$$\alpha = \frac{\delta(1 - \rho)}{1 - \delta\rho}.$$

Therefore, with this approach, the discount factor α depends simultaneously of both, the risk of breakdown $(1 - \rho)$, and the time cost factor δ^2 .

In the present paper we consider a variant of the alternating random-proposer protocol (case (d) of Proposition 9 in Hart and Mas-Colell, 1996), in which all players are equally likely to drop out in case of a breakdown. Calvo (2008) shows that this protocol implements the solidarity value. The main contribution of the present paper is that if we also assume that all players have the same discount factor $\delta \in [0, 1]$, and the risk of breakdown is $(1 - \rho)$, then the subgame perfect equilibrium of the bargaining implements the α -discounted Solidarity value Sl^α , where

$$\alpha = \frac{\delta(1 - \rho)}{1 - \delta\rho},$$

and Sl^α is defined by $Sl^\alpha(N, v) = Sl(N, v_\alpha)$, for all (N, v) .

From the results obtained for the discounted Shapley values and the discounted solidarity values, some consequences of considering the time cost factor in the model can be inferred. Firstly, if players discount payoffs from possible delays in agreements, it may be convenient for game v to be substituted by the discounted game v_α . In addition, it seems evident that for both values, when the risk of breakdown $(1 - \rho)$ and the time discount factor δ are considered simultaneously, the corresponding discounted version of the value immediately arises in a natural way. Finally, for both families obtained, Sh^α and Sl^α , the equal division solution E always appears when players are totally impatient, i.e. when $\delta = 0$. The extreme case of solidarity, given by the equal division value, thus appears when players consider that only immediate agreements are valuable. Hence, strong egalitarianism is supported by strong impatient players.

¹That is, for every period $t = 0, 1, 2, \dots$, the utility of amount x obtained at period t can be represented by $\delta^t x$.

²In a parallel paper, Kawamori (2014) presents a similar variant of the Hart and Mas-Colell model, also considering risk of breakdown and discount factor. Again, the discounted Shapley value is also implemented. The main differences between the two papers is that in our model the proposer offers a payoff allocation for all the members that are still in the game; in Kawamori's protocol the proposer chooses a coalition from among the members still in the game and an allocation for that coalition. On one hand, it gives more freedom to the proposer as she does not need to make an offer to all players, but on the other hand a price must be paid for that freedom as an additional regularity condition is needed in the TU-game in order to guarantee the formation of the full coalition.

The paper is organized as follows. Section 2 presents some preliminary notions on coalitional games and introduces the family of discounted solidarity values. Section 3 describes the non-cooperative bargaining model and shows that the average of the equilibrium proposals is the discounted solidarity value. Section 4 considers the length of the cost delay explicitly, as in Osborne and Rubinstein (1990), and shows the convergence result when the time between rounds vanishes. Section 5 offers axiomatic support for this family. Section 6 considers a more general bargaining procedure, where the discounted Shapley values and the discounted solidarity values appear as particular cases. In there, we characterize the payoffs for each discounted value of this general family. Section 7 presents the final comments.

2 Discounted Solidarity values

Let $U = \{1, 2, \dots\}$ be the (infinite) set of potential players³. We assume that players have preferences over the time when payoffs are reached. In particular we assume that the preferences on $\mathbb{R} \times \{0, 1, 2, 3, \dots\}$ can be represented by the utility function $u(x, t) = \delta^t x$, where the discount factor δ , $0 \leq \delta \leq 1$, is assumed to be equal for all players. In addition, we assume that players are endowed with *von Neumann-Morgenstern preferences* and are *risk-neutral*. Therefore, the expected utility of the lottery $\rho x \oplus (1 - \rho) y$ is $u(\rho x \oplus (1 - \rho) y) = \rho x + (1 - \rho) y$, where the probability of x is ρ , and the probability of y is $(1 - \rho)$.

A *cooperative game* with transferable utility (TU-game) is a pair (N, v) where $N \subset U$ is a non empty and finite set and $v : 2^N \rightarrow \mathbb{R}$ is a *characteristic function*, defined on the power set of N , satisfying $v(\emptyset) = 0$. An element i of N is called a *player* and each non empty subset S of N a *coalition*. The real number $v(S)$ is called the *worth* of coalition S , and is interpreted as the total payoff that the coalition S , if it forms, can obtain for its members. Let G^N denote the set of all cooperative TU-games with player set N . Risk neutral players who use a totally divisible good to make the coalitional payoffs provide an example of such a game. For each $S \subseteq N$, we denote the restriction of (N, v) to S as (S, v) . For the sake of simplicity, we write $S \cup i$ instead of $S \cup \{i\}$, $N \setminus i$ instead of $N \setminus \{i\}$, and $v(i)$ instead of $v(\{i\})$. For each vector $x \in \mathbb{R}^N$, let $x(S) := \sum_{i \in S} x_i$ for each $S \subseteq N$.

A TU-game is said to be *monotonic* if $v(T) \leq v(S)$ whenever $T \subseteq S$. In our setting we made the explicit assumption that the utilities are previously normalized in such a way that when any player leaves the game the payoff that she obtains is *zero*. Monotonicity implies that for any subcoalition S players have incentives to cooperate because every player can attain better payoffs than she could obtain being alone, out of the game. Note also that the payoff $v(i)$ is what player i obtains if the remaining $N \setminus i$ players have left the game, so there is no need for $v(i) = 0$. For example, consider a situation of bankrupt where the liquidation value of one good must be divided between two creditors, i and j , each of which claims the full value. The possible outcomes are that either the good is owned by only one of the players, say i obtains her claim and j receives nothing, i.e. $v(i) = 1$ and j leaves the game receiving zero, or the good is shared between them, i.e. $v(\{i, j\}) = 1$. This is a monotonic game in which players cannot guarantee her $v(i)$. If we want to consider examples, such as market games, where players can guarantee their initial

³From now on, we interpret players in a game as agents with neutral gender. They can be interpreted as automata, institutions, firms, political parties or so on. Therefore we will avoid choosing their gender every time.

endowments without the help of the remaining players, we only need to normalize the utilities such that $v(i) = 0$.

A *value* is a function ψ which assigns to every TU-game (N, v) and every player $i \in N$ a real number $\psi_i(N, v)$, which represents an assessment made by i of its gains from participating in the game. A *payoff configuration* is an element of $\prod_{S \subseteq N} \mathbb{R}^S$.

Let (N, v) be a TU-game. For each coalition $S \subseteq N$ and player $i \in S$, the term $v(S) - v(S \setminus i)$ is the *marginal contribution of player i to coalition S in the TU-game (N, v)* . For each coalition $S \subseteq N$, define

$$\Delta^{av}(v, S) := \sum_{k \in S} \frac{1}{s} (v(S) - v(S \setminus k)).$$

The expression $\Delta^{av}(v, S)$ is the *average of the marginal contributions of players within coalition S in the game (N, v)* .

The *equal split solution E* is defined by

$$E_i(N, v) = \frac{v(N)}{n}, \text{ for all } i \in N.$$

Sprumont (1990; Section 5) introduces the value Sl defined recursively by

$$Sl_i(S, v) = \frac{1}{s} \Delta^{av}(v, S) + \frac{1}{s} \sum_{j \in S \setminus i} Sl_i(S \setminus j, v), \quad (i \in S \subseteq N), \quad (1)$$

starting with

$$Sl_i(\{i\}, v) = v(i), \quad (i \in N).$$

The following formula

$$Sl_i(N, v) = \sum_{\substack{S \subseteq N \\ i \in S}} \frac{(n-s)!(s-1)!}{n!} \Delta^{av}(v, S), \quad (i \in N), \quad (2)$$

was introduced by Nowak and Radzik (1994) in order to define what they called the *Solidarity value* of the game (N, v) . Calvo (2008) shows that definitions (1) and (2) are equivalent.

Given a discount factor $\alpha \in [0, 1]$, define

$$\Delta_{\alpha}^{av}(v, S) := \sum_{k \in S} \frac{1}{s} (v(S) - \alpha v(S \setminus k)).$$

The α -discounted solidarity value Sl^{α} is defined by

$$Sl_i^{\alpha}(N, v) = \sum_{\substack{S \subseteq N \\ i \in S}} \frac{(n-s)!(s-1)!}{n!} \alpha^{n-s} \Delta_{\alpha}^{av}(v, S), \quad (i \in N). \quad (3)$$

Given a game (N, v) and a discount factor $\alpha \in [0, 1]$, the α -discounted game (N, v_{α}) is defined by $v_{\alpha}(S) = \alpha^{n-s} v(S)$, for all $S \subseteq N$. Thus, it turns out that $Sl^{\alpha}(N, v) = Sl(N, v_{\alpha})$.

Alternatively, it is easy to check that Sl^{α} can be defined recursively by

$$Sl_i^{\alpha}(S, v) = \frac{1}{s} \Delta_{\alpha}^{av}(v, S) + \frac{\alpha}{s} \sum_{j \in S \setminus i} Sl_i^{\alpha}(S \setminus j, v), \quad (i \in S \subseteq N), \quad (4)$$

starting with

$$Sl_i^\alpha(\{i\}, v) = v(i), \quad (i \in N).$$

Note also that (4) is equivalent to

$$Sl_i^\alpha(S, v) = \frac{v(S)}{s} + \frac{\alpha}{s} \left[\sum_{j \in S \setminus i} Sl_i^\alpha(S \setminus j, v) - \frac{1}{s} \sum_{k \in S} v(S \setminus k) \right], \quad (i \in S \subseteq N). \quad (5)$$

3 Bargaining with risk of breakdown and time preferences

We model the process that the players must follow to find a cooperative agreement by an *alternating random proposer* protocol. This is a sequential, noncooperative game where the proposer is chosen at random at each step and players drop out of the game randomly after proposals are rejected.

Let $(N, v) \in G^N$ be a TU-game and $0 \leq \rho < 1$ be a fixed parameter:

In each *round* there is a set $S \subseteq N$ of *active* players, and a *proposer* $i \in S$. In the first round the active set is $S = N$. The proposer is chosen at random from S , with all players in S being equally likely to be selected. The proposer makes a feasible offer $a^{S,i} \in \mathbb{R}^S$, i.e. $\sum_{j \in N} a_j^{S,i} \leq v(S)$. If all members of S accept the offer -they are asked in some prespecified order- then the game ends with these payoffs. If the offer is rejected by even one member of S , the players move on to the next round where, the set of active players is again S with probability ρ and a *breakdown* occurs with probability $1 - \rho$: A player j is chosen at random from S to drop out, with all being equally likely to be selected. That player j receives a payoff of zero and the set of active players becomes $S \setminus j$.

This procedure is just case (d) of the general bargaining procedure presented in Section 6 of Hart and Mas-Colell (1996). This bargaining process with three or more players has a broad range of subgame perfect equilibria associated with it. Hence, we consider only *stationary subgame perfect equilibria* (in what follows SP equilibria).

The following result characterizes the offers of an SP equilibrium.

Proposition 1 *Let (N, v) be a monotonic TU-game. Then, there is an SP equilibrium for each specification of the parameters (ρ, δ) . The proposals corresponding to an SP equilibrium are always accepted, and they are characterized by:*

$$(Sl.1) \quad a_i^{S,i}(\rho, \delta) = v(S) - \sum_{j \in S \setminus i} a_j^{S,i}(\rho, \delta) \text{ for each } i \in S \subseteq N; \text{ and}$$

$$(Sl.2) \quad a_j^{S,i}(\rho, \delta) = \delta \left[\rho a_j^S(\rho, \delta) + (1 - \rho) \frac{1}{s} \sum_{k \in S \setminus j} a_j^{S \setminus k}(\rho, \delta) \right] \text{ for each } i, j \in S \text{ with } i \neq j, \text{ and each } S \subseteq N;$$

where $a^S(\rho, \delta) = \frac{1}{s} \sum_{j \in S} a^{S,j}(\rho, \delta)$. Moreover, these proposals are unique and nonnegative.

In other words, (Sl.2) says that i proposes to each $j \in S \setminus i$ the discounted expected payoff that j would get in the continuation of the game in case of rejection; and (Sl.1) says that i gets for himself the remainder up to the full $v(S)$. Note that (Sl.1) and (Sl.2) imply efficiency of the proposals, i.e.

$\sum_{j \in S} a_j^{S,i}(\rho, \delta) = v(S)$, and hence the averages of the proposals are also efficient, i.e. $\sum_{j \in S} a_j^S(\rho, \delta) = v(S)$. Note that every player $j \in S \setminus i$ makes the same proposal to player i . That is, $a_i^{S,j}(\rho, \delta) = a_i^{S,k}(\rho, \delta)$ for every $j, k \in S \setminus i$.

The proof of Proposition 1 follows the same lines used in Proposition 9 of Hart and Mas-Colell (1996). It is only necessary to replace the word "expected payoffs" by "discounted expected payoffs" and the same arguments apply here too. Hence, we skip the proof.

Conditions (Sl.1) and (Sl.2) yield the following recursive formula.

Proposition 2 *Let $(a^S(\rho, \delta))_{S \subseteq N}$ be the average payoff configuration associated with proposals satisfying (Sl.1) and (Sl.2). Then, it holds that*

$$a_i^S(\rho, \delta) = \frac{v(S)}{s} + \frac{1}{s} \frac{\delta(1-\rho)}{1-\delta\rho} \left[\sum_{j \in S \setminus i} a_i^{S,j}(\rho, \delta) - \frac{1}{s} \sum_{k \in S} v(S \setminus k) \right] = St_i^{\alpha(\rho, \delta)}(S, v), \quad i \in S \subseteq N, \quad (6)$$

where $\alpha(\rho, \delta) = \frac{\delta(1-\rho)}{1-\delta\rho}$. Moreover, these vectors $a^S(\rho, \delta)$, $S \subseteq N$, are unique and nonnegative.

Proof. Let (N, v) be a monotonic TU-game. In the following, (ρ, δ) is fixed, so we skip it in the notation. By (Sl.1), for any $i \in S \subseteq N$ it holds that

$$a_i^S = \frac{1}{s} \left(v(S) - \sum_{j \in S \setminus i} a_j^{S,i} \right) + \frac{1}{s} \sum_{j \in S \setminus i} a_i^{S,j}.$$

Applying (Sl.2),

$$\begin{aligned} sa_i^S &= v(S) - \sum_{j \in S \setminus i} \delta \left(\rho a_j^S + (1-\rho) \frac{1}{s} \sum_{k \in S \setminus j} a_j^{S \setminus k} \right) + (s-1) \delta \left(\rho a_i^S + (1-\rho) \frac{1}{s} \sum_{k \in S \setminus i} a_i^{S \setminus k} \right) \\ &= v(S) - \sum_{j \in S \setminus i} \delta \rho a_j^S - \frac{1}{s} \delta (1-\rho) \sum_{j \in S \setminus i} \sum_{k \in S \setminus j} a_j^{S \setminus k} - \delta \rho a_i^S + s \delta \rho a_i^S + (s-1) \frac{1}{s} \delta (1-\rho) \sum_{k \in S \setminus i} a_i^{S \setminus k} \\ &= v(S) - \sum_{j \in S} \delta \rho a_j^S - \frac{1}{s} \delta (1-\rho) \sum_{j \in S} \sum_{k \in S \setminus j} a_j^{S \setminus k} + s \delta \rho a_i^S + \delta (1-\rho) \sum_{k \in S \setminus i} a_i^{S \setminus k} \\ &= (1-\delta\rho) v(S) - \frac{1}{s} \delta (1-\rho) \sum_{k \in S} v(S \setminus k) + s \delta \rho a_i^S + \delta (1-\rho) \sum_{j \in S \setminus i} a_i^{S \setminus j}. \end{aligned}$$

Which finally yields

$$a_i^S = \frac{v(S)}{s} + \frac{1}{s} \frac{\delta(1-\rho)}{1-\delta\rho} \left[\sum_{j \in S \setminus i} a_i^{S \setminus j} - \frac{1}{s} \sum_{k \in S} v(S \setminus k) \right]. \quad (7)$$

The payoffs of the single coalitions $\{i\}$, are $a_i^{\{i\}}(\rho, \delta) = v(i)$, for all $i \in N$. By Monotonicity, it holds that $v(S) - \frac{1}{s} \sum_{k \in S} v(S \setminus k) \geq 0$. Moreover, $\frac{\delta(1-\rho)}{1-\delta\rho} \leq 1$. Therefore, the nonnegativity and the uniqueness of a_i^S follow from the monotonicity of (N, v) and (6) applied recursively.

Finally, comparing (7) with (5), we conclude that $a^S(\rho, \delta) = St^{\alpha(\rho, \delta)}(S, v)$, for all $S \subseteq N$. ■

Notice that (7) is equivalent to

$$a_i^S = \frac{\delta(1-\rho)}{1-\delta\rho} \left[\frac{1}{s} \sum_{j \in S \setminus i} a_i^{S \setminus j} + \frac{1}{s} \sum_{k \in S} \frac{1}{s} [v(S) - v(S \setminus k)] \right] + \frac{1-\delta}{1-\delta\rho} \frac{v(S)}{s}. \quad (8)$$

An interesting property that the SP equilibrium proposals satisfy in the bargaining model associated with the solidarity value is that *being the proposer is always an advantage* in any monotonic game. This is a significant difference with respect to the bargaining protocol associated with the Shapley value, because there it is possible to find examples in which being the proposer can be a disadvantage (see footnote 16 in Hart and Mas-Colell, 1996).

Proposition 3 *Let (N, v) be a monotonic TU-game. Then, for each specification of the parameters (ρ, δ) , for each coalition $S \subseteq N$, and players $i, j \in S$, it holds that $a_i^{S,i}(\rho, \delta) \geq a_i^{S,j}(\rho, \delta)$.*

Proof. Let $i, j \in S \subseteq N$, then

$$\begin{aligned}
a_i^{S,i}(\rho, \delta) - a_i^{S,j}(\rho, \delta) &= v(S) - \sum_{j \in S \setminus i} a_j^{S,i}(\rho, \delta) - \delta \left[\rho a_i^S(\rho, \delta) + (1 - \rho) \frac{1}{s} \sum_{k \in S \setminus i} a_i^{S \setminus k}(\rho, \delta) \right] \quad (9) \\
&= v(S) - \delta \sum_{j \in S \setminus i} \left[\rho a_j^S(\rho, \delta) + (1 - \rho) \frac{1}{s} \sum_{k \in S \setminus j} a_j^{S \setminus k}(\rho, \delta) \right] \\
&\quad - \delta \left[\rho a_i^S(\rho, \delta) + (1 - \rho) \frac{1}{s} \sum_{k \in S \setminus i} a_i^{S \setminus k}(\rho, \delta) \right] \quad (10) \\
&= v(S) - \delta \rho v(S) - \delta (1 - \rho) \frac{1}{s} \sum_{j \in S} \sum_{k \in S \setminus j} a_j^{S \setminus k}(\rho, \delta) \\
&= (1 - \delta \rho) v(S) - \delta (1 - \rho) \frac{1}{s} \sum_{k \in S} v(S \setminus k) \geq (1 - \delta \rho) \left(v(S) - \frac{1}{s} \sum_{k \in S} v(S \setminus k) \right) \geq 0,
\end{aligned}$$

by monotonicity. ■

4 Implementation

In this section we consider the length of the cost delay explicitly, as in Osborne and Rubinstein (1990), and show the convergence result when the time between rounds vanishes.

In the bargaining model in the previous section, bargaining takes place at time $\lambda \in [0, \infty)$ and players have time cost preferences in the form of a time discount factor δ , and a risk of breakdown in the form of a probability $1 - \rho$ in case of the rejection of each offer. Now we make the explicit assumption that both parameters *depend on the time Δ spent between rounds*.

We enumerate the sequence of rounds by $t = 0, 1, 2, \dots$, and denote by $\Delta > 0$ the time that each round takes. It is clear that as the delay between rounds decreases the difference in the valuation of the same amount reached in two consecutive rounds decreases. That is $|u(x, t) - u(x, t + 1)| \rightarrow 0$ when $\Delta \rightarrow 0$. We have assumed that $u(x, t + 1) = \delta u(x, t)$, so it must hold that $\delta \rightarrow 1$. We assume that preferences over time are represented by an exponential discounting function. That is, the present value of x at time τ is $u(x, \tau) = e^{-r\tau}x$, where $r \geq 0$ is the parameter that governs the degree of discounting. For example, in financial economics r is the continuous compound interest rate. Let $\gamma = e^{-\Delta}$, therefore, for each round t , $u(x, t) = e^{-r\Delta t}x = \gamma^{rt}x$. In this case the discount factor is $\delta \equiv \gamma^r$, and when $\Delta \rightarrow 0$ it holds that $\delta \equiv \gamma^r \rightarrow 1$.

On the other hand, we assume that as time goes by the probability of a breakdown of the negotiation increases, with one player living the game. We assume that the rate at which this probability changes over time is a fixed proportion of the probability, that is, $dp(\tau) = -\omega p$. Here, ω is a positive, constant coefficient of proportionality. The negative sign means that $p(\tau)$ is decreasing over time. Taking the initial condition $p(0) = 1$, the solution of this ordinal differential equation is $p(\tau) = e^{-\omega\tau}$. Therefore, $p(t) = e^{-\omega\Delta t}$ is the probability of one player living the game in round t . Denoting $\gamma = e^{-\Delta}$, we have that $p(t) = \gamma^{\omega t}$. Under our stationary assumption this means that after a rejection the probability of being in the game in round t , conditional on still being in the game in round $t - 1$, is $\rho \equiv \gamma^\omega$. That is, $p(t + 1) = p(t)\gamma^\omega$. When the time taken by each round Δ approaches zero, $\rho \equiv \gamma^\omega$ converges to 1.

We now show an explicit formula for the SP equilibrium proposals of Theorem 1 when the time between offers tends to zero, $\Delta \rightarrow 0$, which implies that $\gamma \rightarrow 1$; and when $\Delta \rightarrow \infty$, which implies that $\gamma \rightarrow 0$.

Theorem 4 *Let (N, v) be a monotonic TU-game, and parameters $r, \omega \in \mathbb{R}_{++}$ and $0 \leq \gamma < 1$. Then, for every coalition $S \subseteq N$, and players $i, j \in S$, it holds*

- (1) $\lim_{\gamma \rightarrow 1} (a_i^{S,i}(\gamma) - a_i^{S,j}(\gamma)) = 0$,
- (2) $\lim_{\gamma \rightarrow 1} a_i^S(\gamma) = Sl_i^\alpha(S, v)$, where $\alpha = \frac{\omega}{r+\omega}$,
- (3) $\lim_{\gamma \rightarrow 0} a_i^S(\gamma) = E_i(S, v)$.

Proof. (1) Let $i, j \in S$, by (9), it holds that

$$\lim_{\gamma \rightarrow 1} (a_i^{S,i}(\gamma) - a_i^{S,j}(\gamma)) = \lim_{\gamma \rightarrow 1} \left((1 - \gamma^{r+\omega}) v(S) - \gamma^r (1 - \gamma^\omega) \frac{1}{s} \sum_{k \in S} v(S \setminus k) \right) = 0.$$

(2) The proof is provided by induction. When $S = \{i\}$, it holds that

$$a_i^{\{i\}}(\gamma) = v(i) = Sl_i^\alpha(\{i\}, v).$$

Assume that $a_i^T(1) = Sl_i^\alpha(T, v)$ for each $T \subsetneq S$ and each $i \in T$. For each $i \in S$, following in formula (6), it holds that

$$a_i^S(\gamma) = \frac{v(S)}{s} + \frac{1}{s} \frac{\gamma^r (1 - \gamma^\omega)}{1 - \gamma^{r+\omega}} \left[\sum_{j \in S \setminus i} a_i^{S \setminus j}(\rho, \delta) - \frac{1}{s} \sum_{k \in S} v(S \setminus k) \right]. \quad (11)$$

Applying l'Hôpital's rule, when $\gamma \rightarrow 1$,

$$\lim_{\gamma \rightarrow 1} \frac{\gamma^r (1 - \gamma^\omega)}{1 - \gamma^{r+\omega}} = \lim_{\gamma \rightarrow 1} \frac{r\gamma^{r-1} - (r + \omega)\gamma^{r+\omega-1}}{-(r + \omega)\gamma^{r+\omega-1}} = \frac{\omega}{r + \omega}.$$

Applying the induction hypothesis, $a_i^{S \setminus j}(1) = Sl_i^\alpha(S \setminus j, v)$. Therefore:

$$\lim_{\gamma \rightarrow 1} a_i^S(\gamma) = \frac{v(S)}{s} + \frac{1}{s} \frac{\omega}{r + \omega} \left(\sum_{j \in S \setminus i} Sl_i^\alpha(S \setminus j, v) - \frac{1}{s} \sum_{k \in S} v(S \setminus k) \right) = Sl_i^\alpha(S, v).$$

(3) This is an immediate consequence of making $\gamma \rightarrow 0$ in equation (11). ■

That is:

(1) means that the advantage of being the proposer (see Proposition 3), disappears as the bargaining time Δ decreases, because all equilibrium proposals converge together, and hence converge to the average.

(2) says that these average proposals converge to the α -discounted solidarity values Sl^α , where $\alpha = \frac{\omega}{\omega+r}$.

(3) implies that, independently of the values of the parameters r and ω , when the time Δ tends to ∞ ($\gamma \rightarrow 0$), the value always tends to the equal split solution.

It is clear that $\alpha(r, \omega) = \frac{\omega}{r+\omega}$ is homogeneous with degree zero, i.e. $\alpha(\lambda r, \lambda \omega) = \alpha(r, \omega)$. Therefore, the payoffs of Sl^α are only sensitive to changes in the ratios between parameters r and ω . Recall that when ω decreases the probability $p(t) = e^{-\omega \Delta t}$ of one player leaving the bargaining in round t decreases. On the other hand, when r decreases the time cost factor $e^{-r \Delta t}$ increases. We analyze the behavior of the α -discounted solidarity value Sl^α when the relative importance of the time cost factor (γ^r) and the risk of withdrawing from the bargaining (γ^ω) varies (for any fixed time $\Delta > 0$). The following corollary is an immediate consequence of formula (11).

Corollary 5 (1) *As the risk of breakdown rather than the time cost of bargaining becomes the dominant consideration (r decreases, ω remain fixed) the average of the equilibrium proposals converges to the solidarity value: $\lim_{r \rightarrow 0} Sl^\alpha \rightarrow Sl$.*

(2) *However, if the time cost becomes the dominant consideration (ω decreases, r remain fixed) then the average of the equilibrium proposals converges to the equal split solution: $\lim_{\omega \rightarrow 0} Sl^\alpha \rightarrow E$.*

This result is an extension of the results given in Osborne and Rubinstein (1990) for pure bargaining problems. There, it is shown (see Section 4.5) that when the fear of breakdown rather than the time cost of bargaining is the dominant consideration the equilibrium is close to the Nash bargaining solution, but when the time cost of bargaining rather than the fear of breakdown is the dominant consideration the equilibrium is close to the equal solution. The discounted solidarity value plays the same role in the TU-games setting as the Nash bargaining solution does in pure bargaining games. Notice that both the discounted solidarity value and the discounted Shapley value exhibit the same behavior with respect to r and ω , as proved in Calvo and Gutiérrez-López (2016).

5 Axiomatic characterization

We now provide an axiomatic characterization of the discounted solidarity value.

Two players $\{i, j\} \subseteq N$ are *symmetric* in (N, v) if $v(S \cup i) = v(S \cup j)$, for each $S \subseteq N \setminus \{i, j\}$. Player $i \in N$ is a *null player* in (N, v) if $v(S \cup i) = v(S)$, for each $S \subseteq N \setminus i$. Player $i \in N$ is an *A-null player* in (N, v) if $\Delta^{av}(v, S) = 0$ for all coalitions $S \subseteq N$ containing i .

Consider the following properties of a value ψ in G^N :

Efficiency: For all (N, v) , $\sum_{i \in N} \psi_i(N, v) = v(N)$.

Additivity: For all (N, v) and (N, v') , $\psi(N, v + v') = \psi(N, v) + \psi(N, v')$.

Symmetry: For all (N, v) and all $\{i, j\} \subseteq N$, if i and j are symmetric players in (N, v) then $\psi_i(N, v) = \psi_j(N, v)$.

Null player axiom: For all (N, v) and all $i \in N$, if i is a null player in (N, v) then $\psi_i(N, v) = 0$.

A-Null player axiom: For all (N, v) and all $i \in N$, if i is an A-null player in (N, v) then $\psi_i(N, v) = 0$.

The null player axiom says that if all the marginal contributions of a player in a game are zero (hence it is not a productive player) then it should obtain zero. For the interpretation of the A-null player, notice that $\Delta^{av}(v, S) = 0$ means that the expected productivity of the players in coalition S is zero, because

$$\Delta^{av}(v, S) = \frac{1}{s} \sum_{k \in S} (v(S) - v(S \setminus k))$$

is the expected variation in the worth of coalition S when every player in S has the same chance $1/s$ of leaving the game. The A-null player axiom says that when the average productivity of all coalitions to which player i belongs is zero, then it *must receive zero*.

Nowak and Radzik (1994) offers the following characterization of the solidarity value:

Theorem 6 [Nowak and Radzik, 1994] *A value ψ on G^N satisfies efficiency, additivity, symmetry and the A-null player axiom if, and only if, ψ is the solidarity value.*

Several variants of the A-null player axiom are used in Béal et al. (2017), Casajus and Huetner (2014), Kamijo and Kongo (2012), Nembua (2012), Radzik and Driessen (2016), and Hu and Li (2018).

Notice that the spirit behind the solidarity value is based on a cohesion principle, trying to formulate the idea that all players are "in the same boat". However, this principle is difficult to express in terms of the individual productivity of the players. For that reason, Calvo and Gutiérrez-López (2013) offers an alternative characterization of the solidarity value, with the help of an axiom that makes no reference to the productivity of the players.

Let ψ be a value that determines the player's payoffs, and assume that all players have the same chance of leaving the game. The following expression

$$E[\Delta\psi_i(N, v)] = \frac{1}{n} \sum_{k \in N} (\psi_i(N, v) - \psi_i(N \setminus k, v))$$

yields the *expected variation in the payoffs of player i when every player in coalition N has the same chance of leaving the game* (when player i leaves the game it obtains zero, so we can define $\psi_i(N \setminus i, v) \equiv 0$). Equivalently, $E[\Delta\psi_i(N, v)]$ measures the expected marginal contribution of society (the rest of the players, each with the same probability $1/n$) to the payoffs of player i . We express a cohesion-type rule between players by the equality in the expected marginal contribution of the society to each player:

Definition 1 Equal average gains. *For all (N, v) and all $\{i, j\} \subseteq N$,*

$$E[\Delta\psi_i(N, v)] = E[\Delta\psi_j(N, v)].$$

The following characterization is proved in Calvo and Gutiérrez-López (2013).

Theorem 7 *A value ψ on G satisfies efficiency and equal average gains if, and only if, ψ is the solidarity value.*

We now consider the discounted case. In our setting the parameter $\alpha \in [0, 1]$ determines the discounted payoffs of the players. Assume that initially all players reach the agreement given by $\psi(N, v)$. Now, when

player j leaves the game, in the *next period*, the remaining players can reach the agreement given by $\psi(N \setminus j, v)$. The variation in payoff of player i due to the withdrawing of player j , given the discount factor α , will be $\psi_i(N, v) - \alpha\psi_i(N \setminus j, v)$. Accordingly, we define

$$E_\alpha [\Delta\psi_i(N, v)] = \frac{1}{n} \sum_{k \in N} (\psi_i(N, v) - \alpha\psi_i(N \setminus k, v)).$$

Hence, $E_\alpha [\Delta\psi_i(N, v)]$ is the *expected variation in the discounted payoffs of player i when every player in coalition N has the same probability $\frac{1}{n}$ of leaving the game*. The discounted version of the equal average gains axiom is:

Definition 2 Equal average discounted gains. *For all (N, v) and all $\{i, j\} \subseteq N$,*

$$E_\alpha [\Delta\psi_i(N, v)] = E_\alpha [\Delta\psi_j(N, v)].$$

Recall expression (5), which defines the family of the discounted solidarity values Sl^α . It is immediate that $Sl^\alpha \equiv Sl$ when $\alpha = 1$. Therefore, the equal average discounted gains property reduces to equal average gains, which, jointly with efficiency, characterizes the solidarity value Sl . And when $\alpha = 0$, we have that $Sl^\alpha \equiv E$. In this case, it holds that $E_\alpha [\Delta\psi_i(N, v)] = \psi_i(N, v)$. Now the equal average discounted gains property reduces to the equal payoffs property, which, jointly with efficiency, characterizes the equal split solution E .

We now offer a characterization of the discounted solidarity value with the help of this axiom.

Theorem 8 *Let $\alpha \in [0, 1]$ be the discount factor. A value ψ on G satisfies efficiency and equal average discounted gains if, and only if, ψ is Sl^α .*

Proof. *Existence:* (1) *Efficiency.* We prove that Sl^α satisfies efficiency by induction over $|N|$. If $|N| = 1$, it is immediate as $Sl_i^\alpha(\{i\}, v) = v(i)$. Assume that efficiency holds for $|N| \leq n - 1$. Note that $\Delta_\alpha^{av}(v, N) = v(N) - \frac{1}{n} \sum_{j \in N} \alpha v(N \setminus j)$. Therefore, by (5), it holds that

$$\begin{aligned} \sum_{i \in N} Sl_i^\alpha(N, v) &= \sum_{i \in N} \left[\frac{v(N)}{n} + \frac{\alpha}{n} \left[\sum_{j \in N \setminus i} Sl_i^\alpha(N \setminus j, v) - \frac{1}{n} \sum_{k \in N} v(N \setminus k) \right] \right] \\ &= v(N) + \frac{\alpha}{n} \sum_{i \in N} \left[\sum_{j \in N \setminus i} Sl_i^\alpha(N \setminus j, v) - \frac{1}{n} \sum_{k \in N} v(N \setminus k) \right] \\ &= v(N) + \frac{\alpha}{n} \left[\sum_{j \in N} \sum_{i \in N \setminus j} Sl_i^\alpha(N \setminus j, v) - \sum_{k \in N} v(N \setminus k) \right] = v(N). \end{aligned}$$

(2) *Equal averaged discounted gains.* Since $Sl_i^\alpha(N \setminus i, v) = 0$ and $Sl_j^\alpha(N \setminus j, v) = 0$ then by formula (4), for all $\{i, j\} \subseteq N$, it holds that

$$\begin{aligned} \frac{1}{n} \Delta_\alpha^{av}(v, N) &= Sl_i^\alpha(N, v) - \frac{1}{n} \sum_{k \in N \setminus i} \alpha Sl_i^\alpha(N \setminus k, v) - \frac{1}{n} \alpha Sl_i^\alpha(N \setminus i, v) \\ &= Sl_j^\alpha(N, v) - \frac{1}{n} \sum_{k \in N \setminus j} \alpha Sl_j^\alpha(N \setminus k, v) - \frac{1}{n} \alpha Sl_j^\alpha(N \setminus j, v). \end{aligned}$$

That is,

$$\frac{1}{n} \sum_{k \in N} (Sl_i^\alpha(N, v) - \alpha Sl_i^\alpha(N \setminus k, v)) = \frac{1}{n} \sum_{k \in N} (Sl_j^\alpha(N, v) - \alpha Sl_j^\alpha(N \setminus k, v)).$$

Thus, Sl^α satisfies *equal average discounted gains*.

Uniqueness. Let ψ be a value satisfying the above axioms and let $(N, v) \in G^N$. We prove $\psi = Sl^\alpha$ by induction over the number of players n . If $n = 1$, by *efficiency*, $\psi(\{i\}, v) = Sl_i^\alpha(\{i\}, v) = v(i)$ and hence the result holds. Assume that it is true for fewer than n players. We now prove it for n players.

By *equal average discounted gains*, for all $\{i, j\} \subseteq N$:

$$\frac{1}{n} \sum_{k \in N} (\psi_i(N, v) - \alpha \psi_i(N \setminus k, v)) = \frac{1}{n} \sum_{k \in N} (\psi_j(N, v) - \alpha \psi_j(N \setminus k, v)). \quad (12)$$

By the induction hypothesis, $\psi_i(N \setminus k, v) = Sl_i^\alpha(N \setminus k, v)$, for all $\{i, k\} \subseteq N$. Therefore, following (12):

$$\psi_i(N, v) - \psi_j(N, v) = \alpha \sum_{k \in N} (Sl_i^\alpha(N \setminus k, v) - Sl_j^\alpha(N \setminus k, v)).$$

This expression yields $(n - 1)$ linearly independent equations which, jointly with the *efficiency*,

$$\sum_{i \in N} \psi_i(N, v) = v(N),$$

form an $n \times n$ linear equations system. The matrix of this system is:

$$A_n = \begin{bmatrix} 1 & -1 & 0 & 0 & \dots & 0 \\ 0 & 1 & -1 & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & \ddots & 0 \\ 0 & \dots & \dots & 0 & 1 & -1 \\ 1 & 1 & \dots & 1 & 1 & 1 \end{bmatrix}$$

We now prove that $|A_n| = n$. Indeed, we proceed by induction. For $n = 2$, we have $|A_2| = 2$. Assume that it is true for less than n . We now prove it for n . We develop $|A_n|$ with the elements of the first column:

$$|A_n| = |A_{n-1}| + (-1)^{n-1} \begin{vmatrix} -1 & 0 & 0 & \dots & 0 \\ 1 & -1 & 0 & \dots & 0 \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & 1 & -1 \end{vmatrix} = |A_{n-1}| + (-1)^{n-1} (-1)^{n-1} = n - 1 + 1 = n.$$

Therefore, $|A_n| \neq 0$, which implies that the system has only one solution. Thus, we conclude that $\psi = Sl^\alpha$. ■

6 The general case

Hart and Mas-Colell (1996, Section 6) also considers a more general bargaining procedure in which the proposer is not necessarily the only player to drop out after a proposal rejection. A general value φ is

obtained, being the Shapley and the solidarity values particular specifications of this procedure. In this Section we show that if the players' time discount δ is considered we obtain its corresponding discounted value; and if players become completely impatient, that is $\delta = 0$, a sort of egalitarianism arises.

The general *alternating random proposer* protocol is as follows:

Let $(N, v) \in G^N$ be a TU-game. In each *round* there is a set $S \subseteq N$ of *active* players, and a *proposer* $i \in S$. In the first round the active set is $S = N$. The proposer is chosen at random from S , this is done now according to a given probability distribution $\sigma \in \mathbb{R}_+^S$ (i.e. $\sum_{i \in S} \sigma_i = 1$). The proposer makes a feasible offer $a^{S,i} \in \mathbb{R}^S$, i.e. $\sum_{j \in N} a_j^{S,i} \leq v(S)$. If all members of S accept the offer -they are asked in some prespecified order- then the game ends with these payoffs. If the offer is rejected by even one member of S , the players move on to the next round where, the set of active players is again S with probability $0 \leq \rho_i < 1$ and a *breakdown* occurs with probability $1 - \rho_i$: A player k is chosen at random from S to drop out, with probability $\tau_{k|i}$ (thus $\rho_i + \sum_{k \in S} \tau_{k|i} = 1$, for all $i \in S$). Then, player k receives a payoff of zero and the set of active players becomes $S \setminus k$.

Note that all probabilities above, σ , ρ , and τ , may depend on S and on the proposer.

Now, for each coalition $S \subseteq N$, let us define the following parameters:

$$\begin{aligned} \rho(S) &= \sum_{i \in S} \sigma_i(S) \rho_i(S) , \\ \gamma_{i,k}(S) &= \frac{\sigma_i(S) \tau_{k|i}(S)}{1 - \rho(S)} , \\ \beta_k(S) &= \sum_{i \in S} \gamma_{i,k}(S) . \end{aligned}$$

Thus, $\rho(S)$ is the total probability that set of active players is again S after rejection; $\gamma_{i,k}(S)$ is the conditional probability, given breakdown, that the proposer was i and the dropped out player was k ; and $\beta_k(S)$ is the total probability that k dropped out, given breakdown.

Proposition 9 *Hart and Mas-Colell (1996, proposition 9)* Let (N, v) be a monotonic TU-game. If $\rho(S) < 1$ for all $S \subseteq N$, then there is a unique SP equilibrium, whose payoffs $(a^S)_{S \subseteq N}$ satisfy

$$a_i^S = \sum_{k \in S \setminus i} \beta_k(S) a_i^{S \setminus k} + \sum_{k \in S} \gamma_{i,k}(S) [v(S) - v(S \setminus k)] \quad (13)$$

for all $i \in S \subseteq N$.

The Shapley value corresponds to the case when only the proposer has the chance to drop out after rejection. That is, $\sigma_i(S) = 1/s$, $\rho_i(S) = \rho$, $\tau_{i|i}(S) = 1 - \rho$, and $\tau_{k|i}(S) = 0$ for all $k \neq i$. Then, it holds that $\rho(S) = \rho$, $\gamma_{i,i}(S) = 1/s$, $\gamma_{i,k}(S) = 0$, for all $k \neq i$, and $\beta_k(S) = 1/s$. Therefore, (13) yields the well known recursive formula for the Shapley value

$$a_i^S = \frac{1}{s} \sum_{k \in S \setminus i} a_i^{S \setminus k} + \frac{1}{s} [v(S) - v(S \setminus i)] .$$

The solidarity value corresponds to the case when all players (proposer and respondents) have the same chance to drop out after rejection. That is, $\sigma_i(S) = 1/s$, $\rho_i(S) = \rho$, $\tau_{k|i}(S) = (1 - \rho) \frac{1}{s}$ for all k . Then, it holds that $\rho(S) = \rho$, $\gamma_{i,k}(S) = 1/s^2$, and $\beta_k(S) = 1/s$. Therefore, (13) reduces to

$$a_i^S = \frac{1}{s} \sum_{k \in S \setminus i} a_i^{S \setminus k} + \frac{1}{s} \sum_{k \in S} \frac{1}{s} [v(S) - v(S \setminus k)] . \quad (14)$$

Equation (14) is just equation (1) which defines the solidarity value.

Now we can analyze the effect of introducing the time discount factor δ in the model. Let us define

$$\begin{aligned} \gamma_{i,k}^\delta(S) &= \frac{\delta \sigma_i(S) \tau_{k|i}(S)}{1 - \delta \rho(S)} , \\ \beta_k^\delta(S) &= \sum_{i \in S} \gamma_{i,k}^\delta(S) . \end{aligned}$$

Proposition 10 *Let (N, v) be a monotonic TU-game and a discount factor $0 \leq \delta \leq 1$. Then, there is a unique SP equilibrium, whose payoffs $(a^S)_{S \subseteq N}$ satisfy*

$$a_i^S(\delta) = \sum_{k \in S \setminus i} \beta_k^\delta(S) a_i^{S \setminus k}(\delta) + \sum_{k \in S} \gamma_{i,k}^\delta(S) [v(S) - v(S \setminus k)] + \frac{1 - \delta}{1 - \delta \rho(S)} \sigma_i(S) v(S) \quad (15)$$

for all $i \in S \subseteq N$.

Proof. The proof imitates the arguments of Proposition (1) and Proposition (2) hence, we omit it. In order to obtain (15), note that the proposer $i \in S$ offers to each other player $j \in S \setminus i$ its discounted expected payoff in case of rejection:

$$a_j^{S,i}(\delta) = \rho_i(S) \delta a_j^S(\delta) + \sum_{k \in S} \tau_{k|i}(S) \delta a_j^{S \setminus k}(\delta)$$

and the proposer takes all the surplus, that is

$$\begin{aligned} a_i^{S,i}(\delta) &= v(S) - \sum_{j \in S \setminus i} \left(\rho_i(S) \delta a_j^S(\delta) + \sum_{k \in S} \tau_{k|i}(S) \delta a_j^{S \setminus k}(\delta) \right) \\ &= \delta \rho_i(S) a_i^S(\delta) + (1 - \delta \rho_i(S)) v(S) - \sum_{j \in S \setminus i} \sum_{k \in S} \tau_{k|i}(S) \delta a_j^{S \setminus k}(\delta) . \end{aligned}$$

Therefore, the expected payoff of player i is

$$\begin{aligned} a_i^S(\delta) &= \sigma_i(S) \left(\delta \rho_i(S) a_i^S(\delta) + (1 - \delta \rho_i(S)) v(S) - \delta \sum_{j \in S \setminus i} \sum_{k \in S} \tau_{k|i}(S) a_j^{S \setminus k}(\delta) \right) \\ &\quad + \sum_{j \in S \setminus i} \sigma_j(S) \left(\delta \rho_j(S) a_i^S(\delta) + \delta \sum_{k \in S} \tau_{k|j}(S) a_i^{S \setminus k}(\delta) \right) . \end{aligned}$$

Therefore, we find that

$$(1 - \delta \rho(S)) a_i^S(\delta) = \sigma_i(S) (1 - \delta \rho_i(S)) v(S) - \delta \sigma_i(S) \sum_{j \in S \setminus i} \sum_{k \in S} \tau_{k|i}(S) a_j^{S \setminus k}(\delta) + \delta \sum_{j \in S \setminus i} \sigma_j(S) \sum_{k \in S} \tau_{k|j}(S) a_i^{S \setminus k}(\delta) .$$

As $\sum_{k \in S} \tau_{k|i} = 1 - \rho_i$, we have that

$$\begin{aligned} \sigma_i(S) (1 - \delta \rho_i(S)) v(S) &= \delta \sigma_i(S) (1 - \rho_i(S)) v(S) + \sigma_i(S) (1 - \delta) v(S) \\ &= \delta \sigma_i(S) \sum_{k \in S} \tau_{k|i}(S) v(S) + \sigma_i(S) (1 - \delta) v(S) . \end{aligned}$$

In addition, adding and subtracting the term $\delta\sigma_i(S)\sum_{k\in S}\tau_{k|i}a_i^{S\setminus k}(\delta)$, and dividing by $(1-\delta\rho(S))$, we finally obtain

$$a_i^S(\delta) = \sum_{k\in S\setminus i} \left(\sum_{j\in S} \frac{\delta\sigma_j(S)\tau_{k|j}(S)}{1-\delta\rho(S)} \right) a_i^{S\setminus k}(\delta) + \sum_{k\in S} \frac{\delta\sigma_i(S)\tau_{k|i}(S)}{1-\delta\rho(S)} [v(S) - v(S\setminus k)] + \frac{1-\delta}{1-\delta\rho(S)}\sigma_i(S)v(S)$$

which is equation (15). ■

It is clear that when $\delta = 1$ (15) reduces to (13), and when $\delta = 0$ it holds that $a_i^S(0) = \sigma_i(S)v(S)$. In this case, the payoffs are given by the *weighted split solution* E^σ , defined by

$$E_i^\sigma(S, v) = \sigma_i(S)v(S), \quad (i \in S \subseteq N).$$

Moreover, when $\sigma_i(S) = 1/s$, $\rho_i(S) = \rho$, $\tau_{i|i}(S) = 1 - \rho$, and $\tau_{k|i}(S) = 0$ for all $k \neq i$, we obtain

$$a_i^S(\delta) = \frac{\delta(1-\rho)}{1-\delta\rho} \left[\frac{1}{s} \sum_{j\in S\setminus i} a_i^{S\setminus j}(\delta) + \frac{1}{s} (v(S) - v(S\setminus i)) \right] + \frac{1-\delta}{1-\delta\rho} \frac{v(S)}{s}$$

which define the payoff configuration of the discounted Shapley values (see Calvo and Gutiérrez-López, 2016); and when $\sigma_i(S) = 1/s$, $\rho_i(S) = \rho$, $\tau_{k|i}(S) = (1-\rho)\frac{1}{s}$ for all k , we find that

$$a_i^S(\delta) = \frac{\delta(1-\rho)}{1-\delta\rho} \left[\frac{1}{s} \sum_{j\in S\setminus i} a_i^{S\setminus j}(\delta) + \frac{1}{s} \sum_{k\in S} \frac{1}{s} (v(S) - v(S\setminus k)) \right] + \frac{1-\delta}{1-\delta\rho} \frac{v(S)}{s}$$

which define the payoff configuration of the discounted solidarity values.

7 Final remarks

In this paper we analyze the consequence of introducing the *time cost factor* δ when the solidarity value is considered. In the alternating random-proposal protocol, in which there is a *risk of breakdown* $(1-\rho)$, and all active players are equally likely to drop out of the game in case of a breakdown, the α -discounted solidarity value arises in a natural way. This value is just the solidarity value applied to the α -discounted game. That is, $Sl^\alpha(N, v) = Sl(N, v_\alpha)$, where $v_\alpha(S) = \alpha^{n-s}v(S)$ for all $S \subseteq N$. The discount factor α is determined simultaneously by both the risk of a breakdown and the time cost factor: $\alpha = \frac{\delta(1-\rho)}{1-\delta\rho}$.

The reason why v should be replaced by v_α is quite intuitive. Players are bargaining over agreements that can be delayed in time. Initially, they wish to share the worth $v(N)$, taking into account what each player would obtain eventually if a player i drops out of the game and in the next period players bargain over $v(N\setminus i)$. With von Neumann-Morgenstern preferences and risk-neutral players, this means sharing the expected discounted worth $\alpha v(N\setminus i)$. Again, what they expect to obtain from $v(N\setminus i)$ is conditional on what they would obtain if a new player j drops out of the bargaining in a subsequent period, and they bargain over $v(N\setminus\{i, j\})$, with an expected discounted worth of $\alpha^2 v(N\setminus\{i, j\})$, and so on. Therefore, any coalition $S \subseteq N$ of size $s < n$, needs $n-s$ periods of time, dropping players one by one, to be reached. Hence, the expected discounted worth to be shared is $\alpha^{n-s}v(S)$.

In our context, the extreme case of egalitarianism, given by the equal division value E , appears when players are totally impatient, i.e. when $\delta = 0$ ($\alpha = 0$). In that case, the discounted game (N, v_α) reduces to a pure bargaining game in which only the worth of the grand coalition N matters: $v_0(N) = v(N)$ ⁴, and

⁴Because $0^0 = 1$.

$v_0(S) = 0$, for all $S \subsetneq N$. If players are not discriminated by the bargaining rules, they are symmetric players in v_0 , and hence the expected payoff for each one is $\frac{v(N)}{n}$. Such behavior on the discounted solidarity values Sl^α is also common to the discounted Shapley values Sh^α . They have the equal division value as the extreme case when $\alpha = 0$, that is, $Sl^0(N, v) = Sh^0(N, v) = E(N, v)$.

In summary, when players' time preferences are considered in the alternating random-proposer bargaining model, the impatience of players becomes a source of egalitarianism from a strategic point of view.

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