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>Iván Arribas
ERI-CES, IVIE, University of Valencia

>Amparo Urbano
ERI-CES, University of Valencia

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Identification of efficient equilibria in multiproduct trading with indivisibilities and non-monotonicity[☆]

I. Arribas[a,b,c], A. Urbano[a,c,*]

[a] *Universitat de Valencia, Edificio Departamental Oriental, Campus del Tarongers, 46022 - Valencia, Spain*

[b] *Ivie, Calle Daniel Balaciart, 3 - bajo, 456020 - Valencia, Spain*

[c] *ERI-CES, Universitat de Valencia, c/Serpis 29, 46022 - Valencia, Spain*

Abstract

This paper focuses on multiproduct trading with indivisibilities and where a representative agent may have non-monotonic preferences. In this framework, the set of firms' profits (which comes from *efficient* subgame perfect Nash equilibria) is the Pareto frontier of some projection of the core of the game. We show that under monotonicity efficient subgame perfect Nash equilibria are achieved by single offers and the equilibrium characterization is easy to obtain. When dealing with non-monotonic preferences the problem becomes more challenging. Then, we define a pair of primal-dual linear programming problems that fully identifies the core of the game. A set of modified versions of the dual programming problem characterizes the Pareto-optimal frontier of the core projection on firms' coordinates. Although this approach gives us the payoff-equivalence class (Strong Nash equilibria) of all the efficient subgame perfect Nash equilibria, the number of problems to be solved may be huge.

Keywords: Multiproduct trading, Package assignment problem, Subgame Perfect Nash Equilibrium, Strong Nash Equilibrium,

1. Introduction

In many common situations, agents have complementary preferences for objects in the marketplace. Consider an agent trying to construct a computer system by purchasing components. Among other things, the agent needs to buy a CPU, a keyboard and a monitor, and may have a choice of several models for each component. The agent's valuation of a package depends on

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[*]Communications to: Amparo Urbano, Departamento de Analisis Economico, Edificio departamental oriental, Avenida de los Naranjos s/n, 46022 - Valencia, Spain. Tel.: +34 963 828 207, fax: +34 963 828 249, e-mail: `amparo.urbano@uv.es`

Email addresses: `ivan.arribas@uv.es` (I. Arribas), `amparo.urbano@uv.es` (A. Urbano)

the components in any particular combination, involving products from either only one firm or indeed several firms. This example is a general instance of allocation problems characterized by heterogeneous, discrete resources and complementarities in consumers' preferences. In addition, preferences need not be monotonic. These kinds of models are probably close to many circumstances in real world markets, but they are also more difficult to analyze. On the one hand, with indivisibilities, it is well-known that many familiar properties of the profit functions may fail to ensure the existence of pure strategy Nash equilibrium prices. On the other, without monotonicity of the agent's preferences equilibrium, efficiency may require additional restrictions.

Non-monotonic preferences for bundles of goods have been largely ignored by the current strand of the economic literature. The "more is better" assumption behind monotonicity is a well stablished principle and quite difficult to remove, because, among other things, it makes our lives easier. However, in a world where goods are a collection of attributes, as in Lancaster (1966), the relationship between substitutes and complements will be different from that stemming from the traditional world of monotone preferences. For example, as claimed in Ghaderi *et al.* (2017), whether a color is preferable or not would depend on the red/green coordinate but it is not expected that this attribute is monotone. This approach extends when the buyer consumes a system, made of several simple goods for firms, which is clearly different from the traditional composite good view in the economic literature. In a market for systems, one system can be a substitute for another, yet the components (goods) of each system can be pairwise complements of each other.

This paper focuses on oligopolistic markets in which indivisible goods are sold by multiproduct firms to a continuum of homogeneous buyers, with measure normalized to one, who may have non-monotonic preferences over bundles of products. In these settings, linear pricing does not guarantee the existence of efficient subgame perfect Nash-equilibrium outcomes and, even worse, sometimes equilibrium -either efficient or inefficient- fails to exist (see Liao and Urbano, 2002; Liao and Tauman, 2002; Arribas and Urbano, 2017 -AU hereafter).

AU analyze a kind of non-linear subadditive pricing –*mixed bundling prices*– and analytically show that *efficient* pure strategy subgame perfect Nash equilibria always exist in such settings. Mixed bundling refers to the practice of offering a consumer the option of buying goods separately or else packages of them at a special price. Mixed bundling contracts can be conditional to exclusive dealing for each bundle of two or more goods. Therefore, this kind of contract can be seen as either an aggressive pricing policy for exclusive dealing outcomes or as an out-of-equilibrium offer sustaining the equilibrium consumption sets of individual components in delegated common agency allocations (involving either several firms or all of them). The discrimination on exclusivity helps firms set incentive-compatible contracts by both facilitating collusion on common agency outcomes and by representing a credible threat that avoids deviations by firms. However, the equilibrium need not be unique in the sense that many equilibrium price vectors may sustain the same equilibrium allocation.

The above contracts are in clear contrast with those where firms submit only their consumption contract and the null contract, i.e., make a take-it-or-leave-it (TIOLI) offer. Most of these single offers generally take place under product substitution. To grasp the complexity of non-monotone preferences, we extend the analysis of AU and characterize the efficient equilibrium price vector under monotonicity. This characterization is based on TIOLI contracts. In our first contribution, we offer the conditions guaranteeing TIOLI offers in multiproduct settings, and show that for TIOLI contracts to be equilibrium there is the no need of extended offers. This is the case when the social surplus function is monotonic and exclusive dealing is never an equilibrium consumption set. Monotonicity dramatically simplifies the analysis and the equilibrium price characterization.

Single offers are not possible under non-monotonic preferences and extended contracts are needed. These extended offers are the reason behind the multiplicity of equilibrium price vectors. Moreover, AU also show that inefficient equilibria may also exist where the representative agent chooses a suboptimal bundle and no firm has a profitable deviation inducing the agent to buy the surplus-maximizing bundle because of a coordination problem among firms. Inefficient equilibria can be ruled out by either assuming that all firms (i) price their unsold bundles at the same profit margin as the bundle sold at equilibrium, or (ii) refine the equilibrium correspondence using the solution concept of Strong equilibrium (Aumann, 1959), which requires the absence of profitable deviations by any subset of firms and the agent. Furthermore, AU prove that the set of firms' Strong equilibrium profits is a projection of the core of such games. The above result is important because even theoretically, there are few general results for bundles of more than two goods in strategic settings. In fact, McAdams (McAdams, 1997) found that the analytical machinery for analyzing mixed-bundling could not be easily generalized to even three goods, because of the interaction among sub-bundles.

In spite of AU's theoretical results, when the number of firms is big and/or the number of products of each firm is huge, it is not easy to compute all the efficient subgame perfect equilibria for such games. In fact, even the solution of a simple example of three firms, producing three products each, turns out to be an heroic task, because of sub-bundling interaction and firms' possible deviations. To solve this problem, and as our second contribution, we apply the machinery of the integer programming package problem, or more precisely, the dual problem of its linear relaxation, to *identify* the Pareto-efficient frontier of such games, and hence to find *price vectors satisfying efficient subgame perfection* in a huge set. Specifically, we show that the optimal solutions of the linear relaxation of the package assignment problem give some particular subgame perfect Nash equilibrium. It is interesting to notice that the optimal solutions of any linear programming problem form a polyhedron. In the same fashion, the projection of the dual problem solutions on firms' profit vectors also form a polyhedron, whose Pareto frontier characterizes the set of *all* subgame perfect Nash equilibrium profit vectors belonging to some equivalence class. As this frontier can be expressed as the convex combination of non-dominated Pareto vertices, we only need to obtain all these vertices.

Although integer programming package problems have been used to find Walrasian equilibrium prices (see the related literature section below), their use has not been generalized to identify efficient subgame perfect Nash equilibria. One exception is Arribas and Urbano (2005), who study, as an assignment game, the market interaction of a finite number of single-product firms and a representative buyer, where the buyer consumes bundles of goods. They show that the Nash equilibrium outcomes are solutions for the linear relaxation of an integer programming assignment problem.

As a third contribution, our paper extends the above results to a package assignment problem and illustrates how to modify the dual of the integer programming problem to find the set of efficient subgame perfect Nash equilibria. More specifically, we are interested in the full characterization of the solutions where the firms' profits are non-dominated Pareto. Thus, we want to characterize the polyhedron vertexes of the dual problem optimal solutions whose corresponding coordinates are non Pareto-dominated. The idea is as follows: at each solution where the firms' profits are non-dominated Pareto, there is a set of firms which is better off than under any other solution; among the solutions where this set of firms obtains its highest profits, there is a (second) set of firms which is better off than under any other solution, and so on. We implement this idea by defining a family of dual problems parameterized by an ordered partition of the set of firms. Nevertheless, the number of modified dual problems to be solved may be huge. We also show that when preferences are monotonic the number of modified dual problems is dramatically reduced.

To sum up, we extend first the conditions for TIOLI offers and characterize the equilibrium price vector under monotonicity. Second, we formulate the *efficient* subgame perfect Nash equilibrium outcomes of multiproduct trading with a representative agent as a modified extension of the standard package assignment model. Third, we prove the equivalence of integer programming solutions and efficient (subgame perfect) Nash equilibrium outcomes. Although the results of this paper are driven by the fact that there is a representative buyer (consumers' preferences are homogeneous), our findings open the door for the application of duality methods in more general strategic models.

1.1. Related literature

A large literature (Crawford and Knoer, 1981; Quinzii, 1984; Zhao, 1992; among others) on markets with indivisibilities has grown following Shapley and Shubik's housing market (Shapley and Shubik, 1972). All these works assume price-taking behavior and study the equivalence between core outcomes and those of competitive equilibria. Linear programming has been applied to the standard assignment model, with remarkable results. In particular, there is a linear programming characterization of the standard assignment model yielding optimal solutions to the underlying integer programming problem. The primal and dual solutions of the linear program coincide with price-taking Walrasian equilibrium and with the core. Later authors have addressed variations of the assignment game (Bikhchandani and Mamer, 1997; Ma, 1998; Bikhchandani and Ostroy, 2002). Bikhchandani and Mamer gave a linear programming characterization of the package assignment model, under the usual assumption of linear prices. In an excellent paper, Bikhchandani and Ostroy study assignment problems where individuals trade packages consisting of several, rather than single, objects. Efficient assignments can be formulated as a linear programming problem in which the pricing function expressing duality may be nonlinear in the objects constituting the packages. This extends the price characterization of the standard assignment problem to the package assignment model. The package assignment model was first investigated by Kelso and Crawford (1982), who obtained sufficient conditions for the existence of Walrasian equilibrium. Gul and Stacchetti (1982) obtained equivalent sufficient conditions for the existence of equilibrium and showed that under this condition the core has the lattice property. The point of departure in Bikhchandani and Ostroy is to consider pricing functions which are non-additive over objects and also possibly non-anonymous. More recently, Jaume *et al.* (2012) define a competitive equilibrium for a generalized assignment game and prove its existence by using only linear programming. In particular, they show how to compute equilibrium price vectors from the solution of the dual linear program associated with the primal linear program to find optimal assignments.

The above models consider an economy, with many sellers and many buyers and where all agents are price takers. Thus, prices have to clear the market. Instead, our model deals with many multiproduct firms and a representative agent. The firms post prices for all their feasible bundles and then the agent chooses the priced consumption set that maximizes her surplus. It is then a sequential game dealing with strategic equilibrium, which can also be formulated as a package assignment problem, with the dual problem providing the non-linear prices (when needed). However, the lack of monotonicity of the representative agent's preferences over bundles of products precludes the market clearing condition. In fact, the prices sustaining efficient equilibrium outcomes need not be Walrasian prices, and out-of-equilibrium non-linear prices may be needed to deter firms' deviations and sustain the efficient equilibrium outcome.

A particular relationship between some non-monotonic preferences and linear programming problems has been analyzed by Ghaderi *et al.* (2017), who introduce a new framework for preference disaggregation in multiple criteria decision aiding. The approach aims to infer non-monotonic additive preference models from a set of indirect pairwise comparisons. The preference model is presented as a set of marginal value functions and the discriminatory power of the inferred preference model is maximized against its complexity. To infer a value function that is compatible with the supplied preference information, the proposed methodology leads to a linear programming optimization problem that is easy to solve. The applicability and effectiveness of the new methodology is demonstrated in Ghaderi *et al.* (2014) and Ghaderi *et al.* (2015).

Finally, the paper is also related to the literature in Agency games (see, for example, Bernheim and Whinston (1986a,b)) and, in particular, with studies that explore single contracts (take-it-or-leave-it offers), Peters (2003, 2007).

The paper is organized as follows. The model is presented in Section 2 and the equilibrium analysis in section 3. Section 4 deals with the characterization of TIOLI offers and efficient equilibrium prices under monotonic value functions. Section 5 sets up the Package assignment problem and relates it with the efficient equilibria of our model. Section 6 concludes the paper.

2. The model

Consider a finite set of firms $N = \{1, 2, ..., n\}$ and a continuum of potential homogeneous buyers, with measure normalized to one and common knowledge preferences. Each firm $i \in N$ produces a finite set of heterogeneous goods Ω_i, where each firm's products can be different from or identical to those of any other firm. Let $\Omega = \Omega_1 \times \ldots \times \Omega_n$. Let $c_i(w)$ be the unit cost of production of firm i for good $w \in \Omega_i$, where costs are additive, i.e. $c_i(T) = \sum_{w \in T} c_i(w), T \subseteq \Omega_i$.

A consumption set is a vector $\mathbf{S} = (S_1, \ldots, S_n) \in \Omega$, where $S_i \in 2^{\Omega_i}$ represents firm i selling set S_i in $\mathbf{S}$, which can be the empty set if the agent does not buy anything from firm i. A firm is said to be active in a given consumption set if some of its products are consumed, and non-active otherwise. Let $F(\mathbf{S})$ be the set of active firms in $\mathbf{S}$, i.e. $F(\mathbf{S}) = \{i \in N | S_i \neq \emptyset\}$. Let $c(\mathbf{S}) = \sum_{i \in N} c_i(S_i)$ be the cost of the consumption set $\mathbf{S}$ (where $c_i(\emptyset) = 0$).

A strategy of firm $i \in N$ is a 2^{Ω_i}-tuple specifying the price of each subset of Ω_i. Let $p_i(T)$ be the price of $T \subseteq \Omega_i$ and set $p_i(\emptyset) = 0$. Let $\mathcal{P}_i$ be the set of firm i's strategies, i.e., the set of functions $p_i : 2^{\Omega_i} \longrightarrow \mathbb{R}_+$ and $\mathcal{P} = \mathcal{P}_1 \times \cdots \times \mathcal{P}_n$.

In the first stage, firms choose independently and simultaneously their price schedules $p_i \in \mathcal{P}_i, i \in N$. Then, in the second stage, the agent observes the price vector $\mathbf{p} = (p_1, ..., p_n) \in \mathcal{P}$, and selects a consumption set $\mathbf{S}(\mathbf{p}) \in \Omega$ as a function of $\mathbf{p}$. Thus, the set of agent strategies is the set of functions $\mathbf{S}(\mathbf{p})$ from $\mathcal{P}$ to Ω.

The payoff of each firm $i \in N$ is given by its profit function:

$$\pi_i(\mathbf{S}(\mathbf{p})) = (p_i - c_i)(S_i(\mathbf{p})) \tag{1}$$

where $(p_i - c_i)(S_i(\mathbf{p}))$ means $p_i(S_i(\mathbf{p})) - c_i(S_i(\mathbf{p}))$. When there is no ambiguity we will write $\mathbf{S}$ instead of $\mathbf{S}(\mathbf{p})$, so that firm i's profit is $\pi_i(\mathbf{S}) = (p_i - c_i)(S_i)$.

The representative buyer's payoff function when purchasing $\mathbf{S}$ at prices $\mathbf{p}$ is her consumer surplus,

$$cs[\mathbf{S}, \mathbf{p}] = v(\mathbf{S}(\mathbf{p})) - \sum_{i \in N} p_i(S_i(\mathbf{p})) \tag{2}$$

where function $v : \Omega \longrightarrow \mathbb{R}_+$ denotes the buyer's utility function in monetary terms. This is known by the firms when they decide on the prices of their offerings. In fact, all the above information is common knowledge among the market agents.

Hence, formally, we have a strategic game of complete information with a representative buyer and a set N of firms. Let $G(N + 1, (\Omega_i)_{i \in N}, (\mathcal{P}_i)_{i \in N}, v, c)$ denote such a game.

Definition 1. *A subgame perfect Nash equilibrium (SPE) is a list of strategies,* $(\mathbf{S}, p_1, \ldots, p_n)$ *one for each player, such that:*

$$\mathbf{S} \in \arg\max_{\mathbf{T} \in \Omega} cs[\mathbf{T}, \mathbf{p}] \tag{3}$$

$$p_i \in \arg\max_{p'_i \in \mathcal{P}_i} \pi_i(\mathbf{S}(\mathbf{p}_{-i}, p'_i)) \text{ for all } i \in N \tag{4}$$

where $(\mathbf{p}_{-i}, p'_i) = (p_1, \ldots, p_{i-1}, p'_i, p_{i+1}, \ldots, p_n)$.

Let function $(v-c)(\mathbf{S}) = v(\mathbf{S}) - \sum_{i \in N} c_i(S_i)$ be the *social surplus function*. A set $\mathbf{S}$ is *socially efficient* if $\mathbf{S} \in \arg\max_{\mathbf{T}} (v-c)(\mathbf{T})$. In particular, let $V^* = \max_{\mathbf{T} \in \Omega} : (v-c)(\mathbf{T})$

Notice that the above game with constant costs can be transformed into an equivalent game where production costs are zero. This allows us to simplify the notation and the proof. In particular, given $G(N+1, (\Omega_i)_{i \in N}, (\mathcal{P}_i)_{i \in N}, v, c)$, define the strategic game $G'(N+1, (\Omega_i)_{i \in N}, (\mathcal{P}_i)_{i \in N}, v', c')$, where $v'(\mathbf{S}) = (v-c)(\mathbf{S})$ and $c'_i(w) = 0$ for all $i \in N$, $w \subseteq \Omega_i$. Consider strategies $(\mathbf{S}, p_1, \ldots, p_n)$ and $(\mathbf{S}, p'_1, \ldots, p'_n)$, where for all $\mathbf{S} \in \Omega$ and $i \in N$, $p'_i(S_i) = (p_i - c_i)(S_i)$. It is straightforward to check that $(\mathbf{S}, p_1, \ldots, p_n)$ is a SPE for game G if and only if $(\mathbf{S}, p'_1, \ldots, p'_n)$ is a SPE for game G'. Moreover, the firms' profits and the agent's payoff from both strategies are the same. Given this fact, in what follows we will assume without loss of generality that $c_i(w) = 0$ for all $i \in N$ and $w \subseteq \Omega_i$, i.e., game G has zero production costs.

The use of mixed bundling strategies as opposed to linear prices deserves a comment. Linear prices do not guarantee the existence of (subgame perfect) equilibrium outcomes in our setting, either when the buyer purchases from all firms or when she does so from only one firm (see Liao and Urbano (2002) and AU for technical details). Furthermore, as shown by AU, some subgame perfect Nash equilibrium may not be efficient. To guarantee efficiency, we use the solution concept of subgame perfect Strong Nash equilibrium introduced by Aumann (1959). *Strong equilibrium* is an equilibrium such that no subset of players has a *joint* deviation that strictly benefits *all* of them. Mixed bundling pricing restores equilibrium existence and efficiency. Let SPE^* be the set of *Strong Nash equilibrium* in our game G.

3. Equilibrium analysis

Given a price vector $\mathbf{p} = (p_1, ..., p_n) \in \mathcal{P}_1 \times \cdots \times \mathcal{P}_n$, and $\mathbf{S} \in \Omega$, we define price vector $\mathbf{p}^{\mathbf{S}}$ as,

$$p_i^{\mathbf{S}}(T_i) = \begin{cases} p_i(S_i) & \text{if } i \in F(\mathbf{S}) \\ 0 & \text{if } i \notin F(\mathbf{S}), \end{cases} \tag{5}$$

for all $i \in N$, $T_i \subseteq \Omega_i$. Therefore, each active firm sets prices for its unsold bundles equal to that of its sold bundle and all non-active firms set prices equal to zero.

Theorem 1 in AU proves that the set of subgame perfect Strong Nash equilibrium SPE^* is not empty. More precisely, they show that set $SPE^* = \{(\mathbf{S}, \mathbf{p}) \in SPE | (\mathbf{S}, \mathbf{p}^{\mathbf{S}}) \in SPE\}$. The following Proposition characterizes the set of SPE^*-outcomes of our game summarizing Propositions 1 and 3 in AU. Condition BC below precludes unilateral deviations by the agent.

Condition FC1 guarantees that each active firm individually does not have an incentive to increase the equilibrium prices of its sold bundle (to increase its payoff), since there is at least a bundle where that firm is inactive, which leaves the agent with the same equilibrium payoff. Conditions BC and FC1 are also verified by any SPE-outcome. Finally, since firms are selling bundles of products and at equilibrium they sell at most one bundle, condition FC2 says that no coalition of firms simultaneously has incentives to deviate to increase their profits. Namely, given the other firms' equilibrium strategies: i) active firms in the coalition do not have incentives to set the prices of their unsold bundles in order to sell any of them profitably, and ii) non-active firms cannot benefit from price reductions to zero.

Proposition 1 (AU, 2017). $(\mathbf{S},\mathbf{p})$ *is an* SPE^**-outcome of* G*, where* $\mathbf{S}=(S_1,\ldots,S_n)\in\Omega$ *and* $\mathbf{p}=(p_1,\ldots,p_n)$*,* $p_i\in\mathcal{P}_i$*, with* $p\geq 0$*, if and only if:*

i) For all $\mathbf{T}\in\Omega$*,* $cs[\mathbf{S},\mathbf{p}]\geq cs[\mathbf{T},\mathbf{p}]$*, i.e.,*

$$v(\mathbf{S})-\sum_{i\in F(\mathbf{S})}p_i(S_i)\geq v(\mathbf{T})-\sum_{i\in F(\mathbf{T})}p_i(T_i). \tag{BC}$$

ii) For every $j\in F(\mathbf{S})$ *there is* $\mathbf{T^j}\in\Omega$ *with* $T_j^j=\emptyset$ *such that,* $cs[\mathbf{S},\mathbf{p}]=cs[\mathbf{T^j},\mathbf{p}]$*, i.e.,*

$$v(\mathbf{S})-\sum_{i\in F(\mathbf{S})}p_i(S_i)=v(\mathbf{T}^j)-\sum_{i\in F(\mathbf{T}^j)}p_i(T_i^j). \tag{FC1}$$

iii) For all $A\subseteq N\backslash F(\mathbf{S})$*,* $B\subseteq F(\mathbf{S})$ *and for all* $\mathbf{T}\in\Omega$ *such that* $(A\cup B)\subseteq F(\mathbf{T})$*,*

$cs[\mathbf{S},\mathbf{p}]\geq cs[\mathbf{T},((p_i)_{i\in F(\mathbf{T})\backslash(A\cup B)},(p_i^{\mathbf{S}})_{i\in(A\cup B)})]$*, i.e.,*

$$v(\mathbf{S})-\sum_{i\in F(\mathbf{S})}p_i(S_i)\geq v(\mathbf{T})-\sum_{i\in F(\mathbf{T})\backslash(A\cup B)}p_i(T_i)-\sum_{i\in B}p_i(S_i). \tag{FC2}$$

AU also establish that any SPE^*-consumption set is socially efficient, and conversely that any socially efficient consumption set belongs to an SPE^*-outcome. They also characterize the SPE^*-profit vectors in terms of the core of the game. Let us define the core as:

$$core(v)=\{(\pi^b,(\pi_i)_{i\in N})\in\mathbb{R}_+^{n+1}|\pi^b+\sum_{i\in N}\pi_i=V^* \text{ and } \pi^b+\sum_{i\in F(\mathbf{S})}\pi_i\geq v(\mathbf{S}),\ \forall\mathbf{S}\in\Omega\}.$$

Let Π^{PF} be the Pareto frontier of the projection of $core(v)$ on the n last coordinates and let π^b be the consumer surplus from purchasing $\mathbf{S}$, i.e., $\pi^b = v(\mathbf{S}) - \sum_{i \in F(\mathbf{S})} p_i(S_i)$. Formally,

$$\Pi^{PF} = \quad \{(\pi_i)_{i\in N} \in \mathbb{R}^n_+ | \text{ there is } \pi^b \geq 0 \text{ with } (\pi^b, (\pi_i)_{i\in N}) \in core(v) \text{ and,}$$
$$\text{there is no other } (\pi'^b, (\pi'_i)_{\in N}) \in core(v) \text{ such that}$$
$$\pi'_i \geq \pi_i, \text{ for all } i \in N \text{ with } \pi'_j > \pi_j \text{ for at least some } j\}.$$

The core of the game is defined through linear inequalities and thus, it is a polyhedron and so is Π^{PF}.

There is a map between Π^{PF} and the set of equilibria profit vectors: first, each element $(\pi_i)_{i\in N} \in \Pi^{PF}$ defines an equilibrium price vector such that π_i is firm i's profit; second, if $\mathbf{p}$ is an SPE^*-price vector, then its corresponding firms' profit vector belongs to Π^{PF}. In particular, if $(\pi_i)_{i\in N} \in \Pi^{PF}$, then $(\mathbf{S}, \mathbf{p}) \in SPE^*$, where $\mathbf{S}$ is any socially efficient consumption set and $p_i(T_i) = \pi_i$ for all $i \in N, T_i \subseteq \Omega_i$; conversely, if $(\mathbf{S}, \mathbf{p})$ is an SPE^*-outcome, then $(\pi_i)_{i\in N} \in \Pi^{PF}$, where $\pi_i = p_i(S_i)$, $i \in N$.

4. Equilibria under monotonic value functions and Take-It-or-Leave-It Offers[1]

In Agency Games, competing principals try to control a privately-informed agent's choice. Each principal may offer a single incentive contract that specifies the principal's action as a function of that part of the agent's choice that is contractible by the principal. Well-known Theorems (Peters, 2003, 2007) set up the condition under which the set of pure-strategy equilibrium payoffs in the menu game (where menus of contracts are allowed) coincides with the set of pure strategy equilibrium payoffs in the single contract game in which each principal is allowed to offer only a single contract (i.e., take-it-or-leave-it offer without negotiation, TIOLI). In other words, a condition referred to as the *no-externalities* condition that involves two restrictions. First, each principal's payoff should depend on his own action alone, as well as on the agent's action. However, payoffs should not depend on the actions taken by other principals. Secondly, conditional on her action, the agent's ranking of the actions to be taken by any single principal should be independent of the actions taken by any other principal. This result applies to most of the best known papers on common agency which focus on pure strategy equilibria and assume complete information, as in Bernheim and Whinston (1986a,b).

Peters' no-externalities assumption holds in a Bertrand competition. Once the buyer has chosen to buy from one seller, that seller does not care what prices the other sellers have offered.

[1]We thank an anonymous referee for suggesting this analysis.

In problems like these, menus will only be useful in punishing deviations if the deviation induces the agent to switch her action in a way that hurts the deviator. The above assumption extends to non-linear pricing problems, where many sellers act as principals offering non-linear pricing contracts to the buyer who acts as the agent. A non-linear price is then a contract in which each principal's price can only depend on how much the agent buys from that seller. Conditional on how much to purchase from each principal, the buyer always prefers that a particular principal charges a lower price, no matter what prices the other principals are using.

The analysis of AU extends the insights of delegated agency games under complete information to multiproduct markets with indivisibilities and where the agent's preferences need not be monotone. Single contracts are not possible in our setting and extended contracts are needed[2]. Notice also that in the above Bertrand settings, there is product substitution. In ours, we have to deal with product substitution and complementarity, i.e., some consumption sets can contain complementary products of several firms. This makes Peters's no-externalities assumption difficult to hold. Therefore, we need to look for *new* conditions allowing each principal to offer only a single contract (a TIOLI offer). Consider our game G, where each firm offers any possible bundle of his products and sets a price for each of them, and then the buyer chooses her consumption set. The equilibrium may entail some out-of-equilibrium offers. These out-of-equilibrium prices in G are needed to satisfy condition (FC1) in Proposition 1. This condition states that for each active firm at equilibrium, there is at least one bundle where that firm is non-active that leaves the consumer with the same surplus and prevents an increase of the equilibrium firm prices. Thus, it is needed to set prices for some of the unsold bundles (*out-of-equilibrium prices*) in a specific way. Obviously, for TIOLI contracts we need to get rid of such offers. In our context, there are two situations where out-of-equilibrium prices can be ruled out: first, when the consumer surplus is zero, because no firm can profitably increase the prices of their sold bundles; second, when the out-of-equilibrium prices coincide with the equilibrium prices, i.e., when for each firm j, there is a set $\mathbf{S}^j$, as in condition (FC1), such that $S_i^j = \widetilde{\mathbf{S}}_i$ for all $i \in F(\mathbf{S}^j)$. This result is summarized in the following proposition.

Proposition 2. *Let* $(\mathbf{S}, \mathbf{p})$ *be a SPE*-outcome, then* $(\mathbf{S}, (p_1(S_1), p_2(S_2), \ldots, p_n(S_n)))$ *is a TIOLI SPE*-outcome of* G *(where* $p_i(S_i) = 0$ *if* $i \notin F(\mathbf{S})$*) if and only if one of the following conditions is satisfied:*

i) Consumer surplus is zero, i.e., $v(\mathbf{S}) - \sum_{i=1}^{n} p_i(S_i) = 0$*,*

[2] In a different model, the minimum number of supply schedules (pairs of bundle-price) which supports the equilibrium outcomes has also been studied by other authors (Chiesa and Denicolò, 2009).

ii) For all $j \in F(\mathbf{S})$ there is $\mathbf{S}^j$ with $S^j_j = \emptyset$ and $S^j_i = S_i$ for all $i \in F(\mathbf{S}^j)$, such that $v(\mathbf{S}) - \sum_{i=1}^n p_i(S_i) = v(\mathbf{S}^j) - \sum_{i \in F(\mathbf{S}^j)} p_i(S_i)$

Proof: If $(\mathbf{S}, (p_1(S_1), p_2(S_2), \ldots, p_n(S_n)))$ is a SPE*-outcome, then either i) or ii) are satisfied by (FC1) in Proposition 1.

Now, we have to prove that if $(\mathbf{S}, \mathbf{p})$ is a SPE*-outcome and either i) or ii) are satisfied, then $(\mathbf{S}, (p_1(S_1), p_2(S_2), \ldots, p_n(S_n)))$ is a TIOLI SPE*-outcome. To this end, we need to check that $(\mathbf{S}, (p_1(S_1), p_2(S_2), \ldots, p_n(S_n)))$ satisfies the three conditions in Proposition 1. Conditions (BC) and (FC2) are trivially satisfied because $(\mathbf{S}, \mathbf{p})$ is a SPE*-outcome. Condition (FC1) is verified because condition i) and ii) imply it. ■

Next, we extend AU's analyses to monotonic value functions: v is monotonic if $v(S) \leq v(T)$ for all $S \subseteq T \subseteq \Omega$. We know that inefficient consumption sets that belong to SPE-outcomes might exist. This still verifies under monotonicity as Example 1 below shows. Nevertheless, under *strict* monotonicity Ω is the unique SPE-consumption set and then the unique SPE*-consumption set.

Proposition 3. *If v is strictly monotonic ($v(S) < v(T)$ for all $S \subsetneq T \subseteq \Omega$), then Ω is the unique SPE-consumption set and, consequently, the unique SPE*-consumption set.*

Proof: Let us assume that $(\mathbf{S}, \mathbf{p})$ is a SPE-outcome, with $\mathbf{S} \neq \Omega$. Then there exists $j \in F(\mathbf{S})$ such that $S_j \subsetneq \Omega_j$. Let $S^c_j = \Omega_j \backslash S_j$, which is non-empty.

By FC2 in Proposition 1 with $A = \emptyset$, $B = \{j\}$ and $\mathbf{T} = \mathbf{S} \cup S^c_j$, it is verified that $v(\mathbf{S}) - \sum_{i \in F(\mathbf{S}} p_i(S_i) \geq v(\mathbf{S} \cup S^c_j) - \sum_{i \in F(\mathbf{S}} p_i(S_i)$. Thus, $v(\mathbf{S}) \geq v(\mathbf{S} \cup S^c_j)$, which contradicts that v is strictly monotonic. Therefore, $\mathbf{S} = \Omega$. ■

Example 1: Let the set of firms be $N = \{1, 2\}$, producing $\Omega_1 = \{a, b\}$ and $\Omega_2 = \{c, d\}$. The monotonic buyer's value function is,

$$v(S) = \begin{cases} 0 & |S| = 1 \\ 1 & |S| = 2 \text{ and } \{a,c\} \not\subseteq S \\ 2 & |S| = 3 \text{ and } \{a,c\} \not\subseteq S \\ 3 & |S| \leq 3 \text{ and } \{a,c\} \subseteq S \\ 4 & |S| = 4 \end{cases}$$

It is easy to see that bundle[3] $\{a, c\}$ is an inefficient SPE-consumption set supported by equilibrium prices $p_a = p_c = 1.5$ and $p_b = p_d = p_{ab} = p_{cd} = 4$. The efficient bundle $\{a, b, c, d\}$ is supported as a SPE*-consumption set by equilibrium prices $p_{ab} = p_{cd} = 2$ and $p_a = p_b = p_c = p_d = 1$.

We show next that under monotonicity SPE*-outcomes are TIOLI contracts.

Proposition 4. *Let v be a monotonic value function and let $(\Omega, \mathbf{p}) \in$ SPE*-outcomes. Then $(\Omega, (p_1(\Omega_1), \ldots, p_n(\Omega_n)))$ is a TIOLI SPE*-outcome.*

Proof: If $\sum_{i\in N} p_i(\Omega_i) = v(\Omega)$, then by condition i) in Proposition 2 $(\Omega, (p_1(\Omega_1), \ldots, p_n(\Omega_n)))$ is TIOLI.

Let us assume that $\sum_{i\in N} p_i(\Omega_i) < v(\Omega)$. By condition ii) in Proposition 2, we have to find for all $j \in N$ a set $N^j \subseteq N\backslash j$ such that $v(\Omega) - \sum_{i\in N} p_i(\Omega_i) = v(\bigcup_{i\in N^j} \Omega_i) - \sum_{i\in N^j} p_i(\Omega_i)$.

If $(\Omega, \mathbf{p}) \in$ SPE*-outcomes, then $(\Omega, \mathbf{p}^{\Omega}) \in$ SPE-outcome and has to verify condition FC1 in Proposition 1. Thus, for all $j \in N$ there is $\mathbf{T}^j$ with $T_j^j = \emptyset$, such that $v(\Omega) - \sum_{i\in N} p_i^{\Omega}(\Omega_i) = v(\mathbf{T}^j) - \sum_{i\in F(\mathbf{T}^j)} p_i^{\Omega}(T_i^j)$.

By definition of $\mathbf{p}^{\Omega}$, $p_i^{\Omega}(\Omega_i) = p_i^{\Omega}(T_i^j) = p_i(\Omega_i)$, and by monotonicity $v(\mathbf{T}^j) \leq v(\bigcup_{i\in F(\mathbf{T}^j)} \Omega_i)$. Then, $v(\Omega) - \sum_{i\in N} p_i(\Omega_i) \leq v(\bigcup_{i\in F(\mathbf{T}^j)} \Omega_i) - \sum_{i\in F(\mathbf{T}^j)} p_i(\Omega_i)$.

However, if the above inequality is strict, it contradicts BC in Proposition 1. Therefore, the equality has to hold and then, $N^j = F(\mathbf{T}^j)$ is the searched set. ■

Proposition 4 states that all efficient outcomes can be supported by TIOLI outcomes. Thus, the characterization of SPE*-outcomes simplifies dramatically given that *out-of-equilibrium* offers are not needed to support equilibrium prices. Therefore, both the strategy of a firm and the strategy of the buyer are highly reduced with respect to the non-monotonic case. Now, a strategy of firm i is to set the price of Ω_i, $p_i(\Omega_i) \in \mathbb{R}_+$. Given a price vector, the strategy of the buyer reduces to decide the set of firms from which to buy because each firm only offers a single price-bundle pair. Therefore, the buyer selects a consumption set by choosing a set of firms, $\mathbf{S}(\mathbf{p}) \subseteq N$. However, we can without ambiguity identify a subset A of N with the set of product sold by firms in A, $\bigcup_{i\in A} \Omega_i$. Thus, $(\Omega, \mathbf{p}) \in$ SPE*-outcomes if and only if it verifies BC in Proposition 1 and either i) or ii) in Proposition 2.

To calculate the equilibrium prices we need the following result: under monotonicity, firms'

[3]Notice that a more rigorous notation would be to denote by $\{\{a, b\}, \varnothing\}$ and $\{\varnothing, \{c, d\}\}$ the one-firm bundles and by $\{\{a\}, \{d\}\}$ and $\{\{b\}, \{c\}\}$ the two-firm bundles. To ease the notation and since no confusion will arise we follow the more simple notation of $\{a, b\}$, $\{c, d\}$, $\{a, d\}$ and $\{b, c\}$, to respectively denote such sets.

profits when selling bundle Ω have an upper bound. This bound is their marginal contribution to the market social surplus, as the next lemma states.

Lemma 1. *If* $(\Omega, \mathbf{p}) \in$ *SPE*-outcome set and costs are zero, then* $p_i(\Omega_i) \leq v(\Omega) - v(\Omega \backslash \Omega_i)$ *for all* $i \in N$.

Proof: Given $j \in N$, by condition (BC) in Proposition 1 with $\mathbf{T} = \Omega \backslash \Omega_j$ we obtain that $v(\Omega) - \sum_{i \in N} p_i(\Omega_i) \geq v(\Omega \backslash \Omega_j) - \sum_{i \in N \backslash j} p_i(\Omega_i)$. Therefore, $v(\Omega) - v(\Omega \backslash \Omega_j) \geq p_j(\Omega_j)$. ■

The existence of an upper bound on firms' profits leads us to analyze the conditions that allow all firms (or a subset of them) set the prices (recall that costs are zero) of their products equal to their marginal contributions. Notice that marginal contributions may conflict with firms' maximum profits given that the social surplus has to be non-negative, i.e., $\sum_{i \in N} p_i(\Omega_i) \leq v(\Omega)$, and that marginal contributions do not need to add up to the efficient social surplus (the value of the equilibrium efficient consumption minus costs). We have to analyze, then, what profits firms can reach when either the adding up of marginal contributions is smaller than the efficient social surplus, i.e., $\sum_j [v(\Omega) - v(\Omega \backslash \Omega_j)] \leq v(\Omega)$ or it is higher, i.e., $\sum_j [v(\Omega) - v(\Omega \backslash \Omega_j)] > v(\Omega)$). These questions are studied in what follows. Let us start with the following example.

Example 2: Let the set of firms be $N = \{1, 2, 3\}$, each one producing a single product, thus $\Omega_1 = \{a\}$, $\Omega_2 = \{b\}$, and . $\Omega_3 = \{c\}$. Let us consider three different monotonic value functions (three different consumers' types) as Table 1 shows:

Value functions in Example 2

Value function	{a}	{b}	{c}	{a,b}	{a,c}	{b,c}	{a,b,c}
v_1	1	2	3	7	6	6	8
v_2	1	2	5	7	6	6	8
v_3	1	2	3	7	6	6	10

If the buyer's value function is v_1, the firms' aggregated marginal contributions are lower than the efficient social surplus, i.e., $\sum_i v(\Omega) - v(\Omega \backslash \Omega_j) = 2 + 2 + 1$ is lower that $v(\Omega) = 8$. Moreover, $(\Omega, (p_1(\Omega_1) = 2, p_2(\Omega_2) = 2, p_3(\Omega_3) = 1))$ is a TIOLI SPE*-outcome where firms' profits are their marginal contributions. Therefore, the consumer surplus is positive.

Under value function v_2, the firms' aggregated marginal contributions and the social surplus are the same as under v_1. However, in this case setting prices equal to marginal contributions is not a SPE*-outcome because the buyer will maximize her consumer surplus by buying bundle $\{c\}$ instead of $\{a, b, c\}$. Under v_2, an efficient equilibrium is $(\Omega, (p_1(\Omega_1) = 1, p_2(\Omega_2) = 2, p_3(\Omega_3) =$

1)), where firm 1's profit is lower than its marginal contribution. Therefore, $\sum_i v(\Omega)-v(\Omega\backslash\Omega_j) \leq v(\Omega)$ is not a sufficient condition to guarantee that firms reach their maximum profit.

Finally, when the buyer's value function is v_3, then $\sum_i v(\Omega) - v(\Omega\backslash\Omega_j) = 4 + 4 + 3$ is greater that $v(\Omega) = 10$. Thus not all firms can obtain their marginal contribution. Two efficient equilibria are $(\Omega, (p_1(\Omega_1) = 4, p_2(\Omega_2) = 3, p_3(\Omega_3) = 3))$ and $(\Omega, (p_1(\Omega_1) = 3, p_2(\Omega_2) = 4, p_3(\Omega_3) = 3))$, where firms 2 and 1, respectively, obtain less than their marginal contribution. Moreover, any convex linear combination of these two equilibrium prices is also an equilibrium price. In all those equilibria, firms always extract all consumer surplus.

To reconcile the two principles of marginal contributions and firms' maximum profits, we define next a family of marginal contributions that will characterize the equilibrium prices when firms satisfy some sustitutability conditions.

Let $\mu = \{N_1, N_2, ..., N_L\}$ be an ordered partition of N, i.e., $N_1 \cup \ldots \cup N_L = N$, $N_i \cap N_j = \emptyset$ for $i \neq j$, where the order of the elements in the partition μ is relevant. Thus, $\mu = \{N_1, N_2, N_3\}$ and $\mu' = \{N_2, N_1, N_3\}$ are considered as different partitions, because they give rise to the same partition but with a different order of their elements. Also note that L, the number of sub-sets in the partition, can differ from one partition to the other. Let Γ denote the set of all ordered partitions.

Given an ordered partition $\mu = \{N_1, N_2, ..., N_L\} \in \Gamma$ and a firm $i \in N$, let a_i be the index of the element in the partition μ that includes firm i, $i \in N_{a_i}$; and let P_i^μ be the set of firms which precede and coexists with firm i in partition μ, $P_i^\mu = \bigcup_{l \leq a_i} N_l$.

Definition 2. *The marginal contribution vector $x^\mu(v) \in \mathbb{R}^n$ with respect to v and an ordered partition μ is defined as $x_i^\mu(v) = v(\cup_{j\in P_i^\mu}\Omega_j) - v(\cup_{j\in P_i^\mu}\Omega_j\backslash\Omega_i)$.*

Thus, $x_i^\mu(v)$ is firm i's marginal contribution to the social surplus of a market consisting only of firms in P_i^μ. Firm i, which enters into the market jointly with its peers (firms in N_{a_i}), sets a price for bundle Ω_i equal to its marginal contribution with respect to the coalition of its predecessors and peers. Only the firms in N_L, $x_i^\mu(v) = v(\Omega) - v(\Omega\backslash\Omega_i)$ obtain the whole market marginal contribution.

Following Shapley (1962), we say that firms in set A are *substitutes* if the marginal contribution of set A to the total surplus is higher than or equal to the sum of the individual marginal contributions of firms in A. We extend this concept of substitution to our ordered partitions. Firstly, consider an ordered partition and an arbitrary set of firms in one of its elements (peers), we say that peers' firms in each element of the partition are substitutes when,

i) $\sum_{i \in A}[v(\Omega_{N_1 \cup \ldots \cup N_l}) - v(\Omega_{N_1 \cup \ldots \cup N_l} \backslash \Omega_i)] \leq v(\Omega_{N_1 \cup \ldots \cup N_l}) - v(\Omega_{N_1 \cup \ldots \cup N_l} \backslash \Omega_A)$ for all $l = 1, \ldots, L$ and $A \subseteq N_l$.

That is to say, whenever the increase in the value of the social surplus up to this partition, when this set of peers enters together, is (weakly) higher than when they do so separately.

Secondly, consider now an ordered partition and a set of firms belonging to different elements of that partition. Peers of an element of the partition are substitutes of the peers of the other elements when,

ii) $\sum_{l=1}^{L}[v(\Omega_{N_1 \cup \ldots \cup N_l}) - v(\Omega_{N_1 \cup \ldots \cup N_l} \backslash \Omega_{A \cap N_l})] \leq v(\Omega) - v(\Omega \backslash \Omega_A)$ for all $A \subseteq N$.

In words, whenever the increase in the value of the total social surplus, when these sets of peers enter together, is (weakly) higher than when they do so separately.

The next result gives sufficient conditions to guarantee that for an ordered partition μ, the components of the derived marginal contribution vector $\mathbf{x}^{\mu}(v)$ are the firms' profits of some SPE*-outcomes. Given $A \subseteq N$, let $\Omega_A = \cup_{i \in A} \Omega_i$.

Proposition 5. *Let v be a monotonic value function, suppose that costs are zero and let $\mu = \{N_1, \ldots, N_L\} \in \Gamma$ be an ordered partition such that i) and ii) are satisfied, then $(\Omega, \mathbf{p}) \in$ SPE*-outcomes set, with $p_i(\Omega_i) = x_i^{\mu}(v)$, for all $i \in N$.*

As stated, the above conditions are *sufficient conditions* and there could exist equilibria which do not verify them.

For example, notice that v_2, in Example 2, satisfies the conditions of Proposition 5 for the ordered partition $\{N_1 = \{a\}, N_2 = \{b, c\}\}$. Thus, by the proposition, $(\Omega, \mathbf{p}) \in$ SPE*-outcomes, with $p(a) = x_1^{\mu}(v) = v(\Omega_1) - v(\emptyset) = 1$, $p(b) = x_2^{\mu}(v) = v(\Omega) - v(\Omega \backslash \Omega_2) = 8 - 6 = 2$ and $p(c) = x_3^{\mu}(v) = v(\Omega) - v(\Omega \backslash \Omega_3) = 8 - 7 = 1$. However, also $p(a) = 2$, $p(b) = 1$ and $p(c) = 1$ is a SPE*-outcome, but there is no associated partition with it.

Two cases deserve special consideration, one where there is no partition at all (all firms are peers), $\mu = N$; and another where firms enter one by one in the market (any firm has a peer), $\mu = \{i_1, i_2, \ldots, i_n\}$. In the former, $x_i^{\mu}(v) = v(\Omega) - v(\Omega \backslash \Omega_i)$, which is firm i's marginal contribution to the social surplus of the whole market. In the latter, $x_{i_j}^{\mu}(v) = v(\cup_{k \leq j} \Omega_{i_k}) - v(\cup_{k<j} \Omega_{i_k})$ and only firm i_n obtain his marginal contribution to the whole market. Notice that if $\mu = N$, then the aggregated marginal contributions could exceed the social surplus of the market and firms would not be able to set prices equal to their marginal contributions without additional conditions. On the contrary, if $\mu = \{i_1, i_2, \ldots, i_n\}$, then the aggregated marginal contributions

of the submarkets is equal to the social surplus of the whole market. However, only the last firm entering is able to set a price equal to its contribution to the whole market.

When all firms are peers, $\mu = N$, condition ii) in Proposition 5 is trivially satisfied and condition i) becomes $\sum_{i\in A} v(\Omega) - v(\Omega\backslash\Omega_i) \leq v(\Omega) - v(\Omega\backslash\Omega_A)$ for all $A \subseteq N$. Or, in other words, all firms are substitutes. In this case, Proposition 5 provides the *necessary and sufficient* conditions to guarantee that each firm's profit is equal to its marginal contribution.

Corollary 1. *Let v be a monotonic social surplus function and let costs be zero, then $\sum_{i\in A}(v(\Omega) - v(\Omega\backslash\Omega_i)) \leq v(\Omega) - v(\Omega\backslash\Omega_A)$ is satisfied for all $A \subseteq N$ if and only if $(\Omega, \{p_i(\Omega_i) = v(\Omega) - v(\Omega\backslash\Omega_i)\}_{i\in N}) \in$ SPE*-outcomes.*

Remark 1. *If v is submodular[4], then condition in Corollary 1 is satisfied and for all i, firms i's equilibrium price in any SPS*-outcome is equal to its marginal contribution.*

Notice also that submodularity fulfills the requirements of Peters' *no-externality environments* very precisely: each firm's price (its marginal contribution) is independent of the other firms'prices, the buyer has a weak preference ordering over each firm's prices, and her choice is independent of the other firms' prices for the products.

Now let us consider the second case, where firms enter in the market one by one (any firm has a peer), $\mu = \{i_1, i_2, \ldots, i_n\}$. Condition i) in Proposition 5 is trivially satisfied and condition ii) becomes $\sum_{i_j\in A}(v(\cup_{k\leq j}\Omega_{i_k}) - v(\cup_{k<j}\Omega_{i_k})) \leq v(\Omega) - v(\Omega\backslash\Omega_A)$, for all $A \subseteq N$. Therefore,

Corollary 2. *Let v be a monotonic social surplus function, let costs be zero and let $\mu = \{i_1, i_2, \ldots, i_n\}$ such that $\sum_{i_j\in A}(v(\cup_{k\leq j}\Omega_{i_k}) - v(\cup_{k<j}\Omega_{i_k})) \leq v(\Omega) - v(\Omega\backslash\Omega_A)$ is satisfied for all $A \subseteq N$. Then $(\Omega, \{p_{i_j}(\Omega_{i_j}) = v(\cup_{k\leq j}\Omega_{i_k}) - v(\cup_{k<j}\Omega_{i_k})\}_{i_j\in\mu}) \in$ SPE*-outcomes.*

Remark 2. *If v is supermodular, then any complete partition of N verifies the condition in Corollary 2. Thus, under supermodularity, potentially $n!$ equilibrium price vectors can be defined, one for each ordering of firms.*

Supermodularity reflects complementarities among products or bundles of products and hence among firms. Therefore, it induces only weak market competition so that firms can extract the entire agent surplus. The higher degree of complementary among firms translates

[4]A value function v is submodular if and only if $v(\mathbf{T}+w) - v(\mathbf{T}) \leq v(\mathbf{S}+w) - v(\mathbf{S})$ whenever $\mathbf{S} \subseteq \mathbf{T} \subseteq \Omega\backslash w$. Similarly, v is supermodular if the opposite inequality holds.

into a higher marginal contribution for each of them, so that their sum is bigger than the social surplus. In this framework, firms extract all the social surplus, but not all of them can obtain at equilibrium their social marginal contributions. Now, the sum of firms' marginal contributions exceed the social surplus and each firm's price depends on the other firms' prices, Therefore, this environment is not one of no-externalities, but TIOLI offers will still support pure strategy subgame perfect equilibrium. The reason is that the consumer surplus is zero and thus she is indifferent between buying all products and buying none. Hence, each firm has only to price its own bundle and the null offer.

In Example 2, v_1 satisfies Corollary 1 and v_3 satisfies Corollary 2 for partitions $\{c, b, a\}$ and $\{c, a, b\}$.

To sum up, under monotonicity of the social value function the out-of-equilibrium offers are not needed to sustain the efficient equilibrium outputs, and thus TIOLI offers are sufficient to characterize it. Therefore, monotonicity simplifies the analysis and characterization of both the SPS*-consumption set and the SPS*-price vector set. The first coincides with the set of efficient bundles, those given the same social surplus as Ω. For the second, it is only necessary to set n values, those corresponding to the prices of Ω_i bundles.

If monotonicity is removed and the consumer's surplus is positive, then the out-of-equilibrium prices will play a role in supporting the efficient equilibrium outcome. Therefore, firms will need to set the price of any of their subsets of products. The following example illustrates these results:

Example 3: Let $N = \{1, 2\}$, $\Omega_1 = \{a, b\}$ and $\Omega_2 = \{c, d\}$. The agent value function is:

$$v(S) = \begin{cases} \delta & S = \{a, b\} \\ 9 & S = \{a, d\} \\ 5 & S = \{b, c\} \\ 3 & S = \{c, d\} \\ 0 & \text{otherwise.} \end{cases}$$

By Proposition 1, if $\delta = 2$, then $(\mathbf{S}, \mathbf{p}) \in SPE^*$ if and only if $\mathbf{S} = \{a, d\}$ and $\mathbf{p} = (p_a + p_d = 9, p_a \geq 2, p_d \geq 3; p_{ab} \geq 2, p_{cd} \geq 3; p_b \geq 5 - p_d, p_c \geq 5 - p_a, p_b + p_c \geq 5)$. Thus, the consumer surplus is zero and condition i) in Proposition 2 applies. Therefore, the SPE*-outcome can be implemented by a TIOLI offer: $(\mathbf{S} = \{a, b\}, p_a, p_b)$ is a TIOLI SPE*-outcome, provided that $p_a + p_d = 9$, $p_a \geq 2$ and $p_d \geq 3$. The two last inequalities guarantee that firms do not have any incentive to deviate to their exclusive dealing bundles (pure bundles).

On the other hand, if $\delta = 8$, then $(\mathbf{S}, \mathbf{p}) \in SPE^*$ if and only if $\mathbf{S} = \{a, d\}$ and $\mathbf{p} = (p_a = 6, p_d = 1; p_{ab} \geq 6, p_{cd} \geq 1; p_b \geq 2, p_c \geq 0)$. Now, neither condition i) nor condition ii) of Proposition 2 apply. The consumer surplus is 2 and *out-of-equilibrium-prices* p_{ab} and p_{cd} are needed to prevent deviations. For example, firm 1 cannot profitably increase the price of his product a because, in that case, the consumer will buy bundle $\{c, d\}$, where firm 1's profit is zero, to keep her surplus equal to 2. Therefore, $(\mathbf{S} = \{a, b\}, p_a, p_b)$ is not a TIOLI SPE*-outcome.

In the first case ($\delta = 2$), there is only need to set the prices of two products, a and d; but in the second case ($\delta = 8$), four prices have to be set: two equilibria prices and two out-of-equilibrium-prices.

If the number of firms and products increases, the problem will become harder to analyze and finding equilibrium prices will become quite cumbersome without the use of linear programming tools.

5. The associated package assignment problem

This section focuses on the characterization of the set of SPE^*-outcomes, when the social value function is not monotonic, through a package assignment problem. The main result shows that the set of SPE^*-outcomes is equivalent to integer-valued solutions of the linear relaxation of a package assignment problem.

Since our purpose is to find an assignment of a set of products from different firms to the representative buyer (a package assignment), for any $\mathbf{S} \in \Omega$, define z_S as equal to 1 if the agent chooses consumption set $\mathbf{S}$, and zero otherwise; and for all firm $i \in N$ and set of products $T_i \subseteq \Omega_i$, let $y(T_i, i) = 1$ if firm i sells bundle T_i, and zero otherwise. The integer programming defining the package assignment problem, denoted ILP is,

$$\begin{aligned} V(N) = Max \quad & \sum_{S \in \Omega} v(S) z_S \\ s.t. \quad & \sum_{S \in \Omega} z_S \leq 1 && (6) \\ & \sum_{T_i \subseteq \Omega_i} y(T_i, i) \leq 1 \qquad \forall i \in N && (7) \\ & \sum_{S \ni T_i} z_S \leq y(T_i, i) \qquad \forall i \in N, \forall T_i \subseteq \Omega_i && (8) \\ & z_S,\ y(T_i, i) \in \{0, 1\} \qquad \forall i \in N, \forall T_i \subseteq \Omega_i, \forall S \in \Omega. \end{aligned}$$

Constraint (6) ensures that only one consumption set is selected. Constraints in (7) guarantee that each firm only sells one consumption set, and constraints (8) ensure that firm i sells $T_i \subseteq \Omega_i$ if and only if the selected consumption set S is such that $S_i = T_i$. Constraints (7) and (8) are redundant given the first one, in the sense that they do not reduce the set of feasible solutions. However, they will define the firms' profits and price vectors respectively in the dual problem.

Let us consider the linear relaxation LP of ILP in which we change the integrity constraints z_S, $y(T_i, i) \in \{0, 1\}$ in ILP to $z_S \geq 0$, $y(T_i, i) \geq 0$. Notice that the set of optimal solutions of LP is the set of optimal solutions of ILP. If we remove constraints (7) and (8) of ILP we are left with constraint (6) whose coefficients are equal to 1 and the non-negativity conditions on variables z_S, for all $S \subseteq N$. It is well known that the corner solutions for such a problem are integer: the variable corresponding to a maximum coefficient in the objective function is set to 1 and the remaining variables are set to zero. Hence, in our case, integer solutions always exist in LP and they are the consumption sets S such that $S \in \arg\max_{T \in \Omega} v(T)$. Therefore,

Proposition 6. *The maximum social surplus is equal to the optimal value function of LP,* $V^* = V(N)$. *In other words, if* $sol(LP)$ *is the set of optimal solutions of LP, then* $\arg\max_{T\in\Omega} v(T) = sol(LP)$.

Let DLP be the dual problem of LP. The interest of this formulation is that the dual variable associated with constraint (6) can be interpreted as the consumer surplus, the vector of dual variables associated with constraints (7) can be interpreted as firms' profit vector, and each dual variable associated with each constraint in (8) as the price that firm i sets for each $T_i \subseteq \Omega_i$. Let π^b –the consumer surplus– be the variable associated to constraint (6); let π_i –firm i's profit– be the ones associated to constraints (7) and finally, let $\pi^i_{T_i}$ –price of bundle T_i– be those associated with constraints (8). Then the dual problem DLP is,

$$
\begin{aligned}
Min \quad & \pi^b + \sum_{i\in N} \pi_i \\
s.t. \quad & \pi^b + \sum_{T_i \in S} \pi^i_{T_i} \geq v(S) \qquad \forall S \in \Omega && (9)\\
& \pi_i - \pi^i_{T_i} \geq 0 \qquad \forall i \in N,\ \forall T_i \subseteq \Omega_i && (10)\\
& \pi^b, \pi_i, \pi^i_{T_i} \geq 0.
\end{aligned}
$$

By the fundamental duality theorem (Dantzig, 1974, see pg. 125) if the primal problem has an optimal solution, so does its dual problem and the two optimal value functions are equal. Let sol(DLP) be the set of solutions of DLP and let $(\pi^b, (\pi_i), (\pi^i_{T_i}))$ represent a generic solution of DLP. Notice that if $(\pi^b, (\pi_i), (\pi^i_{T_i})) \in sol$(DLP), then so does $(\pi^b, (\pi_i), (\pi'^i_{T_i}))$, where $\pi^i_{T_i} \leq \pi'^i_{T_i} \leq \pi_i$ for all $i \in N, T_i \subseteq \Omega_i$. Actually, there is always an optimal solution with $\pi'^i_{T_i} = \pi_i$ for all $i \in N, T_i \subseteq \Omega_i$. Hence, the optimal solutions of DLP are such that the buyer and the firms share the maximum social surplus.

However, the solutions of DLP do not necessarily define SPE^*-price vectors because they do not have to maximize the firms' profits. Given $(\pi^b, (\pi_i), (\pi^i_{T_i})) \in sol$(DLP), with $V(N) = \pi^b + \sum_{i \in N} \pi_i$, another optimal solution $(\pi'^b, (\pi'_i), (\pi'^i_{T_i}))$ could exist, such that $V(N) = \pi'^b + \sum_{i \in N}$, with $\pi'_i \geq \pi_i$ for all firms $i \in N$ and $\pi'_j > \pi_j$ for at least some $j \in N$. Thus, the firms' profits in the second solution Pareto dominate those of the first solution. Therefore, firms have incentives to change their price vectors in the first solution to reach the profits of the second one. Let sol^*(DLP) be the sub-set of sol(DLP) such that firms' profits are not Pareto dominated by any other solution of sol(DLP). The next proposition establishes the relationship between the optimal solutions of DLP and the core of the game (proof in the Appendix),

Proposition 7. *Let v be a value function.*

i) If $(\pi^b, (\pi_i), (\pi^i_{T_i})) \in sol(DLP)$, then $(\pi^b, (\pi_i)) \in core(v)$.

ii) if $(\pi^b, (\pi_i)) \in core(v)$, then $(\pi^b, (\pi_i), (\pi^i_{T_i})) \in sol(DLP)$, where $\pi^i_{T_i} = \pi_i$ for all $i \in N$, $T_i \subseteq \Omega_i$.

Proposition 7 states that the sets sol(DLP) and $core(v)$ are equivalent, which yields to the equivalence between the Pareto frontier of those sets with respect to the component $(\pi_i)_{i \in N}$.

Corollary 3. *If $(\pi^b, (\pi_i), (\pi^i_{T_i})) \in sol^*(DLP)$, then $(\pi_i) \in \Pi^{PF}$. The reverse also holds, if $(\pi_i) \in \Pi^{PF}$, then $(\pi^b, (\pi_i), (\pi^i_{T_i})) \in sol^*(DLP)$, where $\pi^i_{T_i} = \pi_i$ and $\pi^b = V(N) - \sum_i \pi_i$.*

Recall that if $(\mathbf{S}, \mathbf{p})$ is an SPE^*-outcome, then so is $(\mathbf{S}, \mathbf{p^S})$ and, clearly, the firms and the agent obtain the same payoffs under such outcomes, i.e., the two equilibria are payoff-equivalent. The binary relation of "*payoff-equivalence*" defines an equivalence relation which provides a partition of the set SPE^* into equivalence classes. Furthermore, $(\mathbf{S}, \mathbf{p^S})$ could be considered as the representative element of its payoff-equivalence class. From this point of view, the linear programming approach provides all the representative elements in the partition of SPE^* with respect to the *payoff-equivalence* relation.

In line with the above reasoning, we prove next that every SPE^*-outcome $(\mathbf{S}, \mathbf{p})$ is such that $\mathbf{S}$ is the solution of the primal problem and $\mathbf{p}$ defines a solution $(\pi^b, (\pi_i), (\pi^i_{T_i})) \in sol^*$(DLP).

The reverse also holds: given both an LP and DLP optimal solutions, with no weakly Pareto dominated component (π_i), they yield an SPE^*-outcome. In other words, the set of firms' profit vectors associated with SPE^*-outcomes is the Pareto frontier of the polyhedron resulting from the projection of sol(DLP) into component (π_i). In the Appendix it is proven.

Proposition 8. *Let v be a value function.*

i) If $\mathbf{S} \in sol(LP)$ and $(\pi^b, (\pi_i), (\pi^i_{T_i})) \in sol^*(DLP)$, then $(\mathbf{S}, \mathbf{p}) \in SPE^*$ where $p_i(T_i) = \pi_i$ for all $i \in N, T_i \in \Omega_i$.

ii) If $(\mathbf{S}, \mathbf{p}) \in SPE^*$, then $\mathbf{S} \in sol(LP)$ and $(\pi^b, (\pi_i), (\pi^i_{T_i})) \in sol^*(DLP)$, where $\pi^b = v(\mathbf{S}) - \sum_{i \in F(\mathbf{S})} p_i^{\mathbf{S}}(S_i)$, $\pi_i = \pi^i_{T_i} = p_i(S_i)$ for all $i \in N, T_i \in \Omega_i$.

Notice that i) in Proposition 8 is also proof of the existence of SPE^*-outcomes.

The question that remains is how to characterize the set sol^*(DLP). To this end, we have to modify the above DLP to guarantee that the optimal solutions of the new linear programming problem define a non Pareto-dominated set of firms' profit vectors. A direct way to do so consists of changing the objective function to $\max \sum_{i \in N} \pi_i$ and adding the constraint $V(N) = \pi^b + \sum_{i \in N} \pi_i$ to guarantee that the solutions still belong to $sol(DLP)$.[5]

However, we are interested in the full characterization of the solutions where the firms' profits are non-dominated Pareto. Therefore, we want to characterize the polyhedron vertexes of the DLP optimal solutions whose coordinates (π_i) are non Pareto-dominated. First, since the value of $V(N)$ is known, constraint $\pi^b + \sum_{i \in N} \pi_i = V(N)$ in the DLP guarantees that the feasible solutions of this new problem are the optimal ones for DLP. Second, at each solution where the firms' profits are non-dominated Pareto, there is a set of firms which is better off than under any other solution; among these solutions where this set of firms obtain its highest profits, there is a (second) set of firms which is better off than under any other; and so on. Thus, we define a family of problems parameterized by an ordered partition of the set of firms. More precisely, let $\mu = \{N_1, N_2, ..., N_L\}$ be an ordered partition of N, i.e., $N_1 \cup \ldots \cup N_L = N$, $N_i \cap N_j = \emptyset$ for $i \neq j$, and the order of the elements in the partition μ is relevant: the first element of the partition is N_1, the second is N_2 and the last one is N_L. Thus, $\mu = \{N_1, N_2, N_3\}$ and $\mu' = \{N_2, N_1, N_3\}$ will be considered as different because they give rise to the same partition but with a different order in their elements. Also note that L, the number of sub-sets in the partition, can differ from one partition to another. Let Γ denote the set of all ordered partitions.

[5]This is not the only way to do it and there are other possible objective functions which identify non Pareto-dominated set of firms' profits.

Given an ordered partition $\mu \in \Gamma$, define μ-DLP as:

$$
\begin{aligned}
Max \qquad & \sum_{l=1}^{L} \left(\sum_{i \in N_l} \pi_i \right) 10^{d(L-l)} \\
s.t. \qquad & \pi^b + \sum_{T_i \in S} \pi^i_{T_i} \geq v(S) \qquad \forall S \in \Omega && (11) \\
& \pi_i - \pi^i_{T_i} \geq 0 \qquad \forall i \in N, \ \forall T_i \subseteq \Omega_i && (12) \\
& \pi^b + \sum_{i \in N} \pi_i = V(N) && (13) \\
& \pi^b, \pi_i, \pi^i_{T_i} \geq 0,
\end{aligned}
$$

where d is any integer number such that $n \cdot \lceil V(N) \rceil < 10^d$, where $\lceil . \rceil$ is the ceiling function that maps a real number to the following (smallest) integer number.

It is straightforward to realize that any feasible solution of μ-DLP is an optimal solution of DLP. The constraints in the partition formulation do not change the set of optimal solutions of DLP but the objective function chooses a particular one among them. The objective function is a number for which each set of d consecutive digits are determined by $\sum_{i \in N_l} \pi_i$. Thus, the first d digits are *occupied* by $\sum_{i \in N_1} \pi_i$, the second set of d digits by $\sum_{i \in N_2} \pi_i$ and so on. Hence, an optimal solution of $sol(\mu$-DLP) gives one of the most preferred price vectors by firms in N_1; it gives one of the most preferred price vectors by firms in N_2, among those most preferred by firms in N_1; and so on.

Therefore, the optimal solutions of μ-DLP are those of DLP that are not Pareto-dominated with respect to the variables $(\pi_i)_{i \in N}$, In this way, $sol(\mu$-DLP) $\subseteq sol^*$(DLP) $\subseteq sol$(DLP), as the next Proposition states (see proof in the Appendix).

Proposition 9. *For any $\mu \in \Gamma$ it is verified that $sol(\mu$-DLP$) \subseteq sol^*(DLP)$.*

From Proposition 9 and Proposition 8.ii it is straightforward that an optimal solution of LP and μ-DLP is an SPE^*-outcome of G (Corollary 4), i.e., sol(LP) gives the equilibrium consumption set and $sol(\mu$-DLP) the equilibrium profits and price vectors, for some partition μ.

Corollary 4. *Let v be a value function, $\mu \in \Gamma$, $\mathbf{S} \in sol(LP)$ and $(\pi^b, (\pi_i), (\pi^i_{T_i})) \in sol(\mu$-DLP$)$. Then, $(S, p) \in SPE^*$, where $\pi^i_{T_i} \leq p_i(T_i) \leq \pi_i$ for all $i \in N$ and $T_i \subseteq \Omega_i$, $T_i \neq S_i$, and $p_i(S_i) = \pi_i$.*

Example 3 (continuation): Recall that $(\mathbf{S},\mathbf{p}) \in SPE^*$ if and only if $\mathbf{S} = \{a,d\}$ and $\mathbf{p} = (p_a + p_d = 9, p_a \geq 2, p_d \geq 3; p_{ab} \geq 2, p_{cd} \geq 3; p_b \geq 5 - p_d, p_c \geq 5 - p_a, p_b + p_c \geq 5)$ for $\delta = 2$; and $\mathbf{p} = (p_a = 6, p_d = 1; p_{ab} \geq 6, p_{cd} \geq 1; p_b \geq 2, p_c \geq 0)$, for $\delta = 8$.

On the other hand, solving the primal linear problem we find that $sol(\text{LP}) = \{a,d\}$. The solutions of the modified dual problem for partitions $\mu_1 = \{\{1\},\{2\}\}$ and $\mu_2 = \{\{2\},\{1\}\}$ are shown in Table 2.

Table 1: solutions of μ-DLP for $\delta = 2, 8$ and $\mu = \{\{1\},\{2\}\}$ and $\{\{2\},\{1\}\}$.

δ	μ	π^b	π_1	π_2	π_a^1	π_b^1	π_{ab}^1	π_c^2	π_d^2	π_{cd}^2
2	$\{\{1\},\{2\}\}$	0	6	3	6	2	2	3	3	3
2	$\{\{2\},\{1\}\}$	0	2	7	2	2	2	3	7	3
8	$\{\{1\},\{2\}\}$	2	6	1	6	2	6	0	1	1
8	$\{\{2\},\{1\}\}$	2	6	1	6	2	6	0	1	1

Therefore, for $\delta = 2$, by Proposition 8 $(\mathbf{S},\mathbf{p}_i) \in SPE^*$, $i = 1, 2$ where $\mathbf{S} = \{a,d\}$ and

$$\begin{aligned}
\mathbf{p}_1 &= (p_a = 6, 2 \leq p_b \leq 6, 2 \leq p_{ab} \leq 6; p_c = 3, p_d = 3, p_{cd} = 3) \\
\mathbf{p}_2 &= (p_a = 2, p_b = 2, p_{ab} = 2; 3 \leq p_c \leq 7, p_d = 7, 3 \leq p_{cd} \leq 7).
\end{aligned}$$

Since the consumer surplus is zero, then condition i) in Proposition 2 applies. Thus, as we already saw, this SPE*-outcome can be implemented by a TIOLI offer: $(\mathbf{S} = \{a,b\}, p_a, p_b)$, provided that $p_a + p_d = 9$, $p_a \geq 2$ and $p_d \geq 3$.

When $\delta = 8$, partition does not play any role and linear programming gives the unique outcome $\mathbf{S} = \{a,d\}$ and $\mathbf{p} = (p_a = 6, 2 \leq p_b \leq 6, p_{ab} = 6; 0 \leq p_c \leq 1, p_d = 1, p_{cd} = 1)$.

Notice that in both cases the prices provided by the linear programming approach are included in those provided by the characterization of SPE^* in Proposition 1. However, as example 3 shows, the reverse does not need to be satisfied, and there are SPE^*-prices which are not a solution of the modified dual problem. There are two main reasons for this fact: on the one hand, the prices coming from the modified dual problem are upper bounded by firms' profits, while this does not happen for those prices characterized by Proposition 1. On the other hand, each solution to a modified dual problem corresponds to a vertex of the optimal firms' profit polyhedron. Meanwhile, Proposition 1 characterizes the complete polyhedron.

At a first glance, we see an inordinate number of linear programming problems to be solved to characterize efficient Strong Nash subgame perfect equilibria. This brings up the natural question: what is to be gained by the linear programming approach that is not already present in AU's characterization?[6] We could provide two answers at least. Firstly, the AU's characterization is not operative. For example, it implies solving a system with a huge number of inequalities. Second, the linear programming approach allows us to isolate some precise equilibria, which we may be interested in, for instance, those giving more profits to some specific sets of firms. This may be particularly important in some markets, where the size of the firms could be a proxy for their competitive advantage, and, therefore, the ordered partition in the dual programming problem should mimic this market ordering by size. As another example, niche businesses specialize in a certain area or type of product, often allowing them a competitive advantage over other businesses. Small business owners often function as niche companies, targeting a specific industry or type of consumer. It could be interesting to analyze the price setting and profits of this set of firms. Summing up, the LP approach helps to carry out a variety of studies, by partitioning the set of firms in specific clusters.

For completeness, we state now the results already known for monotonic social surplus functions, in terms of the solutions of μ-DLP. Suppose that the social value function is submodular, then, by Proposition 8, set sol^*(DLP) has only a single element which is characterized by solving the modified dual problem for the partition $\mu = N$. This result is stated in the following Proposition.

Proposition 10. *Let v be a submodular value function, then $sol^*(DLP) = sol(N\text{-}DLP)$*

Next, suppose that the monotonic social value function is supermodular instead. Then, the vector of the social surplus increments is a corner in the polyhedron of the set of firms' equilibrium profit. Thus, different orders in the set of firms generate different corners, and the convex hull of these corners is also a feasible equilibrium profit. Therefore, each permutation of the set of firms defines a corner in the polyhedron of the set of firms' equilibrium profits. Formally, let Σ be the set of permutations of firms (permutation of N) and let $\sigma \in \Sigma$ be any of its elements. Then sol^*(DLP) has (at most) as many vertexes as permutations for the number of firms, i.e., factorial of n.

Proposition 11. *Let v be a supermodular value function, then $sol^*(DLP) = conv\big\{sol(\sigma\text{-}DLP) | \sigma \in \Sigma\big\}$.*

[6]We thank an anonymous referee for pointing out this drawback.

6. Conclusion

This paper focuses on oligopolistic markets in which indivisible goods are sold by multiproduct firms to a continuum of homogeneous buyers, with measure normalized to one, who may have non-monotonic preferences over bundles of products.

To better explain the complexity of non-monotonic preferences, we offered first the conditions guaranteeing single offers (take-it-or-leave-it offers, TIOLI) in multiproduct settings. Then, we show that under monotonicity efficient subgame perfect Nash equilibria are achieved by TIOLI offers, and easily characterized efficient equilibrium prices.

Turning back to non-monotonic preferences, we applied the machinery of the integer programming package problem, or more precisely, the dual problem of its linear relaxation, to *identify* the Pareto-efficient frontier of such games and hence to find all price vectors satisfying efficient subgame perfection in a huge set. The optimal solutions of any linear programming problem form a polyhedron, and so does the projection of the dual problem solutions on firms' profit vectors. We identified the Pareto frontier of the above projection in order to characterize the set of all subgame perfect Nash equilibrium profit vectors belonging to some equivalence class. These payoff-equivalence classes come from subgame perfect Strong Nash equilibrium outcomes.

Our paper illustrates how to modify the dual of the integer programming problem to find subgame perfect Strong Nash equilibrium outcomes. More specifically, since we were interested in the full characterization of the solutions, where the firms' profits are non-dominated Pareto, we characterized the polyhedron vertexes of the dual problem optimal solutions whose corresponding coordinates are non-dominated Pareto. The idea is as follows: at each solution where the firms' profits are non-dominated Pareto, there is a set of firms where firms are better off than under any other solution; among the solutions where this set of firms obtains their highest profits, there is a (second) set of firms where firms are better off than under any other solution, and so on. We implement this idea by defining a family of dual problems parameterized by an ordered partition of the set of firms. Nevertheless, the number of modified dual problems to be solved may be huge. We also showed that when preferences are monotonic the number of modified dual problems is drastically reduced.

An interesting extension of our analysis is to consider both multiple firms and multiple agents. Our intuition is that a new kind of non-linear pricing (*non-anonymous*prices) is needed to ensure the existence of (subgame perfect) Nash equilibrium. The characterization of these equilibria will require a new approach to tackle associate integer programming problems. This is left for future research.

7. Appendix

Proof of Proposition 7 Let $(\pi^b, (\pi_i), (\pi^i_{T_i})) \in sol(\text{DLP})$. Constraints (9) and (10) imply $\pi^b + \sum_{i \in F(\mathbf{S})} \pi_i \geq v(\mathbf{S})$ for all $\mathbf{S} \in \Omega$; and by optimality and Proposition 6 $\pi^b + \sum_{i\in N} \pi_i = V(N) = V^*$. Therefore, $(\pi^b, (\pi_i)) \in core(v)$.

Now let $(\pi^b, (\pi_i)) \in core(v)$ and define $(\pi^b, (\pi_i), (\pi^i_{T_i}))$, where $\pi^i_{T_i} = \pi_i$, so that it verifies contraint (10). Moreover, given $\mathbf{S}$, then $\pi^b + \sum_{T_i \in \mathbf{S}} \pi^i_{T_i} = \pi^b + \sum_{i\in F(\mathbf{S})} \pi_i \geq v(\mathbf{S})$. Therefore, $(\pi^b, (\pi_i), (\pi^i_{T_i}))$ is a feasible solution of DLP. However, $\pi^b + \sum_{i\in N} \pi_i = V(N)$, and thus it is also an optimal solution of DLP, $(\pi^b, (\pi_i), (\pi^i_{T_i})) \in sol(\text{DLP})$.

■

Before proving Proposition 8, let us show Lemma 2, which states that the optimal solutions of the LP and DLP problems set the prices of non-active firms equal to marginal costs, i.e., equal to zero (property ii) and the profits of any active firm bigger than or equal to their selling prices (property i).

Lemma 2. *Let* $\mathbf{S} \in sol(LP)$ *and* $(\pi^b, (\pi_i), (\pi^i_{T_i})) \in sol(DLP)$, *then*

i) $\pi_i = \pi^i_{S_i}$ *and* $\pi_i \geq \pi^i_{T_i}$ *for all* $i \in F(\mathbf{S}), T_i \subseteq \Omega_i$

ii) $\pi_i = \pi^i_{T_i} = 0$ *for all* $i \notin F(\mathbf{S})$ *and* $T_i \subseteq \Omega_i$.

Proof: First we prove ii). If $i \notin F(\mathbf{S})$, then constraint $\sum_{T_i \subseteq \Omega_i} y(T, i) \leq 1$ is satisfied with strict inequality and by the complementary slackness condition $\pi_i = 0$. Now, by constraint (10) in DLP, $0 = \pi_i \geq \pi^i_{T_i} \geq 0$ and ii) is satisfied.

Now we prove i). Given that $z_{\mathbf{S}} = 1$, then by the complementary slackness condition, $\pi^b + \sum_{S_i \in \mathbf{S}} \pi^i_{S_i} = v(\mathbf{S})$ and because the optimal value of DLP is equal to the optimal value of LP, then $\pi^b + \sum_{i\in N} \pi_i = v(\mathbf{S})$. Thus, $v(\mathbf{S}) - \sum_{i \in N} \pi_i = v(\mathbf{S}) - \sum_{S_i \in \mathbf{S}} \pi^i_{S_i}$, which, by ii) already proven, implies that $\sum_{S_i \in \mathbf{S}} \pi^i_{S_i} = \sum_{i\in F(\mathbf{S})} \pi_i$. This, in turn, implies that for all $i \in F(\mathbf{S})$, $\pi_i = \pi^i_{S_i}$ given that by constraints (10) $\pi_i \geq \pi^i_{S_i}$. ■

Proof of Proposition 8

First we prove i). Let $\mathbf{S} \in sol(\text{LP})$ and $(\pi^b, (\pi_i), (\pi^i_{T_i})) \in sol^*(\text{DLP})$, and define $p_i(T_i) = \pi_i$ for all $i \in N, T_i \in \Omega_i$. Let us see that $(\mathbf{S}, \mathbf{p}) \in SPE^*$.

Sept 1. Condition BC: Given $\mathbf{T} \in \Omega$, by constraint (9), $\pi^b + \sum_{T_i \in \mathbf{T}} \pi^i_{T_i} \geq v(\mathbf{T})$ which, by constraint (10), implies that $\pi^b + \sum_{i\in F(\mathbf{T})} \pi_i \geq v(\mathbf{T})$. We have defined $p_i(T_i) = \pi_i$, thus $\pi^b \geq v(\mathbf{T}) - \sum_{i\in F(\mathbf{T})} p_i(T_i)$. On the other hand, we have that $\pi^b + \sum_{i\in F(\mathbf{S})} \pi_i = v(\mathbf{S})$ and therefore, $\pi^b = v(\mathbf{S}) - \sum_{i\in F(\mathbf{S})} p_i(S_i)$. Thus, $v(\mathbf{S}) - \sum_{i\in F(\mathbf{S})} p_i(S_i) \geq v(\mathbf{T}) - \sum_{i\in F(\mathbf{T})} p_i(T_i)$ which is condition BC.

Step 2. Condition FC1: If $\pi^b = 0$, then FC1 holds for $\mathbf{S}^j = \emptyset$. If $\pi^b > 0$, then suppose that there exits $j \in F(\mathbf{S})$ such that for all $\mathbf{T} \in \Omega$ with $T_j = \emptyset$, it is verified that $v(\mathbf{S}) - \sum_{i\in F(\mathbf{S})} p_i(S_i) > v(\mathbf{T}) - \sum_{i\in F(\mathbf{T})} p_i(T_i)$.

Let $\epsilon = \frac{1}{2}\min_{\mathbf{T}\in\Omega, T_j=\emptyset}\{v(\mathbf{S}) - \sum_{i\in F(\mathbf{S})} p_i(S_i) - (v(\mathbf{T}) - \sum_{i\in F(\mathbf{T})} p_i(T_i))\} > 0$. Define,

$$\begin{aligned}
\widehat{\pi}^b &= \pi^b - \epsilon \\
\widehat{\pi}_i &= \begin{cases} \pi_i + \epsilon & if \quad i = j \\ \pi_i & if \quad i \neq j \end{cases} \\
\widehat{\pi}^i_{T_i} &= \begin{cases} \pi^i_{T_i} + \epsilon & if \quad i = j, T_i = S_i \\ \pi^i_{T_i} & \quad otherwise. \end{cases}
\end{aligned}$$

Let us show that $(\widehat{\pi}^b, (\widehat{\pi}_i), (\widehat{\pi}^i_{T_i}))$ satisfies all DLP constraints. Given $\mathbf{T} \in \Omega$, if $T_j = S_j$ then,

$$\widehat{\pi}^b + \sum_{T_i\in\mathbf{T}} \widehat{\pi}^i_{T_i} = \pi^b - \epsilon + \sum_{T_i\in\mathbf{T}} \pi^i_{T_i} + \epsilon = \pi^b + \sum_{T_i\in\mathbf{T}} \pi^i_{T_i} \geq v(\mathbf{T}).$$

Otherwise,

$$\begin{aligned}
\widehat{\pi}^b + \sum_{T_i\in\mathbf{T}} \widehat{\pi}^i_{T_i} &= \pi^b - \epsilon + \sum_{T_i\in\mathbf{T}} \pi^i_{T_i} \\
&\geq \pi^b - \frac{1}{2}\Big(v(\mathbf{S}) - \sum_{i\in F(\mathbf{S})} p_i(S_i) - v(\mathbf{T}) + \sum_{i\in F(\mathbf{T})} p_i(T_i)\Big) + \sum_{T_i\in\mathbf{T}} \pi^i_{T_i} \\
&= \pi^b - \frac{1}{2}\Big(v(\mathbf{S}) - \sum_{S_i\in\mathbf{S}} \pi^i_{S_i} - v(\mathbf{T}) + \sum_{T_i\in\mathbf{T}} \pi^i_{T_i}\Big) + \sum_{T_i\in\mathbf{T}} \pi^i_{T_i} \\
&= \pi^b - \frac{1}{2}\Big(\pi^b - v(\mathbf{T}) + \sum_{T_i\in\mathbf{T}} \pi^i_{T_i}\Big) + \sum_{T_i\in\mathbf{T}} \pi^i_{T_i} \\
&= \frac{1}{2}\Big(v(\mathbf{T}) + \pi^b + \sum_{T_i\in\mathbf{T}} \pi^i_{T_i}\Big) \geq \frac{1}{2}(v(\mathbf{T}) + v(\mathbf{T})) = v(\mathbf{T}).
\end{aligned}$$

Thus, constraint (9) is satisfied. To prove that it also satisfies constraint (10), consider different cases. For all $i \neq j$, $\widehat{\pi}_i - \widehat{\pi}^i_{T_i} = \pi_i - \pi^i_{T_i} \geq 0$. If $i = j$ but $T_i \neq S_i$, then $\widehat{\pi}_i - \widehat{\pi}^i_{T_i} = \pi_i + \epsilon - \pi^i_{T_i} \geq \pi_i - \pi^i_{T_i} \geq 0$; if $i = j$ and $T_i = S_i$, then $\widehat{\pi}_i - \widehat{\pi}^i_{T_i} = \pi_i + \epsilon - (\pi^i_{T_i} + \epsilon) = \pi_i - \pi^i_{T_i} \geq 0$.

Finally, $\widehat{\pi}^b + \sum_{i\in N} \widehat{\pi}_i = \pi^b - \epsilon + \sum_{i\in N} \pi_i + \epsilon = \pi^b + \sum_{i\in N} \pi_i = V(N)$. Then $(\widehat{\pi}^b, (\widehat{\pi}_i), (\widehat{\pi}^i_{T_i}))$ verifies all the constraints of the DLP problem. However, $\widehat{\pi}_i \geq \pi_i$ for all $i \neq j$ with $\widehat{\pi}_j > \pi_j$ which contradicts that $(\pi^b, (\pi_i), (\pi^i_{T_i})) \in sol^*(\text{DLP})$, i.e., the Pareto dominance of the component (π_i). Hence FC1 is verified.

Step 3. Condition FC2: Let $A \subseteq N \backslash F(\mathbf{S})$, $B \subseteq F(\mathbf{S})$. By Lemma 2, $\pi_i = \pi^i_{T_i} = 0$ for all $i \in A, T_i \subseteq \Omega_i$; and $\pi_i = \pi^i_{\mathbf{S}_i} \geq \pi^i_{T_i}$ for all $i \in B, T_i \subseteq \Omega_i$. Then, given $\mathbf{T} \in \Omega$, where $(A \cup B) \subseteq F(T)$,

$$\begin{aligned}
v(\mathbf{S}) - \sum_{i\in F(\mathbf{S})} p_i(S_i) &= v(\mathbf{S}) - \sum_{S_i\in\mathbf{S}} \pi^i_{S_i} \\
&\geq v(\mathbf{T}) - \sum_{T_i\in\mathbf{T}} \pi^i_{T_i} \\
&= v(\mathbf{T}) - \sum_{\substack{T_i\in\mathbf{T} \\ i\notin(A\cup B)}} \pi^i_{T_i} - \sum_{\substack{T_i\in\mathbf{T} \\ i\in B}} \pi^i_{T_i} \\
&\geq v(\mathbf{T}) - \sum_{i\in F(\mathbf{T})\backslash(A\cup B)} p_i(T_i) - \sum_{i\in B} p_i(S_i),
\end{aligned}$$

and FC2 is verified.

Thus, $(\mathbf{S}, \mathbf{p})$ verifies conditions BC, FC1 and FC2, and hence $(\mathbf{S}, \mathbf{p}) \in SPE^*$.

Now we prove ii). Let $(\mathbf{S}, \mathbf{p}) \in SPE^*$. First we prove that $\mathbf{S} \in sol(\text{LP})$. If $(\mathbf{S}, \mathbf{p})$ is an SPE^*-outcome, then $(\mathbf{S}, \mathbf{p}^{\mathbf{S}}) \in SPE^*$. By BC in Proposition 1, $cs[\mathbf{S}, \mathbf{p}^{\mathbf{S}}] \geq cs[\mathbf{S}', \mathbf{p}^{\mathbf{S}}]$ for all $\mathbf{S}' \in \Omega$. Therefore,

$$\begin{aligned} v(\mathbf{S}) - v(\mathbf{S}') &\geq \sum_{i \in F(\mathbf{S})} p_i^{\mathbf{S}}(S_i) - \sum_{i \in F(\mathbf{S}')} p_i^{\mathbf{S}}(S_i') \\ &= \sum_{i \in F(\mathbf{S})} p_i^{\mathbf{S}}(S_i) - \sum_{i \in F(\mathbf{S}) \cap F(\mathbf{S}')} p_i^{\mathbf{S}}(S_i') \\ &= \sum_{i \in F(\mathbf{S}) \setminus F(\mathbf{S}')} p_i(S_i) \geq 0, \end{aligned}$$

given that $p_i^{\mathbf{S}}(S_i') = 0$ for all $i \notin F(\mathbf{S})$ and $p_i^{\mathbf{S}}(S_i') = p_i^{\mathbf{S}}(S_i)$ for all $i \in F(\mathbf{S}) \cap F(\mathbf{S}')$. Thus, $v(\mathbf{S}) \geq v(\mathbf{S}')$ for every $\mathbf{S}' \in \Omega$.

Second, we prove that $(\pi^b, (\pi_i), (\pi^i_{T_i})) \in sol^*(\text{DLP})$, where $\pi^b = v(\mathbf{S}) - \sum_{i \in N} p_i^{\mathbf{S}}(S_i)$, $\pi_i = p_i^{\mathbf{S}}(S_i)$ and $\pi^i_{T_i} = p_i^{\mathbf{S}}(T_i)$ for all $i \in N, T_i \in \Omega_i$. Constraint (9) is equivalent to condition BC and constraint (10) is verified because by definition $p_i^{\mathbf{S}}(S_i) \geq p_i^{\mathbf{S}}(T_i)$. Moreover, $\pi^b + \sum_i \pi_i = v(\mathbf{S})$, the optimal solution of the primal problem, thus $(\pi^b, (\pi_i), (\pi^i_{T_i})) \in sol(\text{DLP})$. Now suppose that $(\pi^b, (\pi_i), (\pi^i_{T_i})) \notin sol^*(\text{DLP})$, then there exists another solution of $sol^*(\text{DLP})$ that Pareto dominates component (π_i). But then, by (i), already proven, we can construct an SPE^*-outcome with a profit vector that Pareto dominates the profits in $(\mathbf{S}, \mathbf{p})$ which contradicts that $(\mathbf{S}, \mathbf{p}) \in SPE^*$. Therefore, $(\pi^b, (\pi_i), (\pi^i_{T_i})) \in sol^*(\text{DLP})$. ■

Proof of Proposition 9

Given $\mu = (N_1, \ldots, N_l) \in \Gamma$, let $(\pi^b, (\pi_i), (\pi^i_{T_i})) \in sol(\mu\text{-DLP})$. Let us assume that $(\pi^b, (\pi_i), (\pi^i_{T_i})) \notin sol^*(\text{DLP})$, so there exists $(\widehat{\pi}^b, (\widehat{\pi}_i), (\widehat{\pi}^i_{T_i})) \in sol^*(\text{DLP})$ such that for all $i \in N$ $\widehat{\pi}_i \geq \pi_i$ and $\widehat{\pi}_j \geq \pi_j$ for at least one $j \in N$. Then there is a $k \leq l$ such that $\sum_{i \in N_k} \widehat{\pi}_i > \sum_{i \in N_k} \pi_i$, and $\sum_{i \in N_m} \widehat{\pi}_i \geq \sum_{i \in N_m} \pi_i$ for all $m \leq l$ and $m \neq k$. This implies that $\sum_{l=1}^{L} \left(\sum_{i \in N_l} \widehat{\pi}_i \right) 10^{d(L-l)} > \sum_{l=1}^{L} \left(\sum_{i \in N_l} \pi_i \right) 10^{d(L-l)}$ which contradicts that $(\pi^b, (\pi_i), (\pi^i_{T_i})) \in sol(\mu\text{-DLP})$. Therefore, $(\pi^b, (\pi_i), (\pi^i_{T_i})) \in sol^*(\text{DLP})$. ■

Proof of Proposition 10

By Proposition 9 $sol(\text{N-DLP}) \subseteq sol^*(\text{DLP})$, thus it suffices to prove that $sol^*(\text{DLP}) \subseteq sol(\text{N-DLP})$.

Let $(\widehat{\pi}^b, (\widehat{\pi}_i), (\widehat{\pi}^i_{T_i})) \in sol^*(\text{DLP})$. If $(\pi^b, (\pi_i), (\pi^i_{T_i})) \in sol(\text{N-DLP})$, then $\sum_i \pi_i \geq \sum_i \widehat{\pi}_i$.

By monotonicity of v and by Proposition 8, $(\mathbf{\Omega}, \widehat{\mathbf{p}}) \in SPE^*$, where $\widehat{p}_i(\Omega_i) = \widehat{\pi}_i$. Moreover, by Corollary 1, $\widehat{p}_i(\Omega_i) = v(\Omega) - v(\Omega \setminus \Omega_i)$ for all $i \in N$, so that $\widehat{\pi}_i = v(\Omega) - v(\Omega \setminus \Omega_i)$. On the other hand, also $(\mathbf{\Omega}, \mathbf{p}) \in SPE^*$, where $p_i(\Omega_i) = \pi_i$ and by Lemma 1, $p_i(\Omega_i) \leq v(\Omega) - v(\Omega \setminus \Omega_i)$, so that $\pi_i \leq v(\Omega) - v(\Omega \setminus \Omega_i)$. Therefore, $\pi_i \leq \widehat{\pi}_i$ and $\sum_i \pi_i \leq \sum_i \widehat{\pi}_i$.

Thus, we have proven that $\sum_i \pi_i = \sum_i \widehat{\pi}_i$, which implies that $(\widehat{\pi}^b, (\widehat{\pi}_i), (\widehat{\pi}^i_{T_i})) \in sol(\text{N-DLP})$. ■

Proof of Proposition 11.

Step 1: $sol^*(\text{DLP}) \supseteq conv\{sol(\sigma\text{-DLP}) | \sigma \in \Sigma\}$.

Given σ and by Proposition 9, $sol(\sigma\text{-DLP}) \subseteq sol^*(\text{DLP})$ and by Corollary 2, $\sum_i \pi_i = V(N)$. Thus, independently of σ, the sum of the firms' profits of any solution in $sol(\sigma\text{-DLP})$ is constant and equal to the social surplus. Therefore, $conv\{sol(\sigma\text{-DLP})|\sigma \in \Sigma\} \subseteq sol^*(\text{DLP})$.

Step 2: $sol^*(\text{DLP}) \subseteq conv\{sol(\sigma\text{-DLP})|\sigma \in \Sigma\}$.

By Proposition 8, any element $(\pi^b, (\pi_i), (\pi^i_{T_i})) \in sol^*(\text{DLP})$ defines a SPE^*-outcome that, by Corollary 2, has the firms' profit vector in $conv\{x^\sigma(v)|\sigma \in \Sigma\}$ and the consumer surplus is equal to zero. Therefore, $(\pi_i) \in conv\{x^\sigma(v)|\sigma \in \Sigma\}$ and $\pi^b = 0$.

On the other hand, given σ, let P_i^σ be the set of firms which precede firm i with respect to permutation σ, i.e. for all $i \in N$, $P_i^\sigma = \{j \in N|\sigma(j) < \sigma(i)\}$. Define, the marginal contribution vector $x^\sigma(v) \in R^n$ of v with respect to σ by, $x_i^\sigma(v) = v(\cup_{j\in\{P_i^\sigma+i\}} \Omega_j) - v(\cup_{j\in P_i^\sigma} \Omega_j))$, for all $i \in N$. If v is supermodular, then the marginal contribution vector $x^\sigma(v)$ is positive. Define $\pi^b = 0$, $\pi_i = x_i^\sigma(v)$ and $\pi^i_{T_i} = \pi_i$. It is a straightforward exercise to see that $(\pi^b, (\pi_i), (\pi^i_{T_i})) \in sol(\sigma\text{-DLP})$. Thus, any element $(\pi_i) \in conv\{x^\sigma(v)|\sigma \in \Sigma\}$ defines a vector $(0, (\pi_i), (\pi^i_{T_i})) \in conv\{sol(\sigma\text{-DLP})|\sigma \in \Sigma\}$.

Putting together the results of the two steps above, we conclude that $sol^*(\text{DLP}) \subseteq conv\{sol(\sigma\text{-DLP})|\sigma \in \Sigma\}$.
■

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