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TAX FEDERALISM AND COOPERATIVE GAMES: VALUE APPROACH

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Abstract
We model the problem of how to distribute public spending among the different regions of a country once all taxes are collected. We model the problem as a cooperative game in coalitional form. A tax game is built, specifying how much tax is collected in every region and coalition of regions in the country under secession. In this paper we propose two tax rules: the *balanced tax rule*, and the *p-balanced tax rule*. Both rules have the property of being stable for every tax problem, as they belong to the core of the tax game. The Spanish case is considered as example. We compare their redistributive behavior with the present Spanish financial system, with the population egalitarian, and with the optimistic secession tax rules.

Keywords: fiscal federalism; fiscal stability; secessionism; coalitional games; Shapley value.

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1 INTRODUCTION

This is the second of two papers devoted to the study of tax allocations from a cooperative game theory perspective. We refer to the introduction of the first paper, Calvo (2018), for the motivation and interpretation of a regional financial system as a cooperative tax game.

A cooperative *tax game* concerns the distribution of the total public budget among central government and regional authorities, once all taxes have been collected in the country. The player set is formed by the regions of the country; and the characteristic function of the game specifies how much total collected taxes would be at the disposal of each region or set of regions under secession.

A *tax allocation* specifies how much of the budget will be spent in each region of the country. Insofar, as the tax game is estimated, we can determine the set of *stable tax allocations*. That is, the set of tax budget distribution among regions that cannot be objected to by any coalition of regions, in the sense that they could obtain a greater tax amount on their own than the allocation gives to them. This is the familiar concept of the core of a cooperative game. A tax financial rule will be stable if the allocated grants among regions belong to the core of the tax game.

The main result obtained in Calvo (2018) is of a positive nature: *the set of stable tax allocations is always nonempty*. The source of this result is related with the familiar notion of increasing returns to scale. In our tax context this property says that larger coalitions of regions have a relatively larger tax budget to share among them. The translation of this property into the setting of cooperative games is the notion of *convex games*. This is the key fact which guarantees the existence of stable tax allocations, as every convex game has a non-empty core. This fact has a clear consequence of a political nature: it is always possible to design a *stable* financing regional system.

Therefore, the analytic tool of tax games can be used to evaluate the stability of the present financing system in a country. In Calvo (2018), the Spanish case was considered as an example, showing that the present financial system was unstable from 2011 to 2014.

Alternatively, we can check the stability of any new proposal for modification in the financial system. For example, a rule which has wide support, especially on the left of the political spectrum, is the *population egalitarian rule*. Its purpose is that every citizen in a country should not be discriminated against in public spending. In particular, there should be no disparities in public spending per capita between the regions of a country. Unfortunately, if there are wide per capita rent disparities among regions, it follows that the same per capita expenses in every region could yield an unstable tax allocation. Spain is a case of wide disparities in terms of per capita rent across regions. As we can expect, we showed in our first paper that the population egalitarian rule applied in Spain also yields an unstable tax allocation.

The motivation of the present paper is to go a step further, building tax rules which are stable. Indeed, we present a tax rule which is called the *balanced tax rule* φ. The basic idea in its design is to balance the contribution of welfare that each region makes to each other in form of taxes. The computation of φ is rather simple, and we calculate it for the Spanish case.

We can observe that when there are wide disparities in the per capita wealth between regions, φ usually exhibits a poor redistributive rent behavior. We note this fact for the Spanish case. Therefore, we propose a *p-balanced tax* rule φ^p, where the weights p are the per capita rent of regions. By its construction, this rule coincides with the weighted Shapley value (Shapley, 1953a) and we show that, for the Spanish case, it offers a greater degree of solidarity between regions than φ.

The paper is organized as follows: Section 2 contains the basic model and summarizes the main stability results obtained in Calvo (2018). We build a *tax game* (N, v_t^α), where N is the set of regions of the country, and $v_t^\alpha(S)$ is the total tax budget at disposal of each possible coalition of regions $S \subseteq N$. The worth $v_t^\alpha(S)$ is the sum of all collected taxes in each region of S, derived from trade between regions of S and with foreign countries. These are the total collected taxes that the coalition of regions S can manage by themselves. The parameter α measures the degree of trade relationships that remains between regions S and $N \backslash S$ after this secessionist process, where $\alpha = 0$ is the most pessimistic case, and $\alpha = 1$ the most optimistic one. A tax allocation will be a way to share the total collected taxes between the regions of N. We define the *set of stable tax allocations* of a tax problem as the *core of its associated tax game*, denoted by $C(N, v_t^\alpha)$. By construction, it turns out that the tax game is *convex*. And this implies that the *set of stable tax allocations is always non empty* for every value of α.

In Section 3, we present a tax rule which *is placed in the center of the core of the pessimistic tax game* v_t^0. We call it the *balanced tax rule* φ. The basic idea in its design is to balance the contribution of the welfare that each region makes to each other in the form of taxes. Its computation is very easy: Firstly, the tax rule φ gives to each region i all direct and indirect taxes collected within region, as a result of its internal commercial activity, its exports, and its imports from abroad. Secondly, the additional amount of taxes collected due to the relationship between region i and every $j \neq i$ resulting from their bilateral trade is equally distributed between them.

If we are worried about the solidarity between rich and poor regions, we need to refine the balanced tax rule incorporating the redistribution principle. For this reason, we propose a weighted version of the balanced tax rule. The idea is to take into account the relative richness level between each pair of regions $i, j \in N$ when they redistribute their tax benefits shared *inversely proportional* to their per capita richness level. We prove that φ^p *is also stable in the pessimistic tax game*. That is, no matter how different the per capita richness level between regions is, the redistributive effect of the tax rule always yields a stable outcome. Hence, we can see φ^p of a plausible trade-off between the principles of stability and solidarity.

We also define the *secessionist tax rule* ζ^α which computes how much tax every would-be independent region would expect to manage, for every scenario α. We call ζ^1 the optimistic secessionist tax rule. We realize that ζ^1 is stable for any scenario α.

Finally, we end this Section with a numerical example for a better understanding of all these concepts.

In Section 4, we compute the tax rules φ and φ^p for the Spanish case. We also compare these tax rules with the present tax rule followed in Spain, the population egalitarian rule, and the optimistic secession rule ζ^1.

In this work we identify the set of stable tax allocations of a tax problem (N, T) by the set of stable allocations of its associated tax game (N, v_t^α). In the literature regarding cooperative games theory, several single-valued solutions had been defined. Therefore, an obvious question that immediately arises is the translation of solutions defined in the setting of cooperative games into the setting of tax problems, by using its associated tax game. In Section 5, we show that three different solutions: the *Shapley value* (Shapley, 1953a,b), the *nucleolus* (Schmeidler, 1969), and the τ-value (Tijs, 1981), when they are applied to the pessimistic tax game, all of them coincide with the balanced tax rule φ. In the Appendix we provide proof of this technical result, as well as a brief revision of the Literature regarding this equivalence result.

2 THE TAX MODEL

In this Section we offer a summary of the tax model presented. This allows us to make a self-contained exposition in the present paper. In Calvo (2018, Sections 2 and 3), the reader will find a comprehensive explanation and justification of the notions presented here, with all corresponding references.

Let $N = \{1,2,\dots,n\}$ be the set of regions in a country. The impact that trade among regions has in tax collection is summarized by the *tax matrix* $T = [t_{ij}]$, where we assume $t_{ij} \geq 0$ for all $i,j \in N$[2]. The total amount $T(N)$ is the sum of all taxes collected within the country from all regions.

For each region in a country, the total taxes collected are determined by the economic activity within the region, trade with other regions, and the imports and exports from abroad. Usually the data of the national account are aggregate at most by regions. Then we know for each region i, the aggregate $T_i(N)$, which represents all taxes collected. This amount is the sum of direct taxes $D_i(N)$ plus indirect taxes $I_i(N)$. Hence, if we wish to obtain a disaggregated account vector $T_i = (t_{i1},\dots,t_{in})$ we need to make and indirect estimation. To this end we use the trade matrix $C = [c_{ij}]$ which measures the flow of goods and services between the regions of a country. That is, for each pair of regions $i,j \in N$, c_{ij} will be the amount of goods and services of region i sold to region j. Additionally, we denote by m_i total foreign imports of region i, and by x_i total foreign exports.

Direct taxes are determined by the production of goods and services sold within the region, by sales to other regions, and by exports abroad. Therefore, the disaggregation of these taxes among regions is made by:

$$\begin{cases} D_{ii} = \dfrac{c_{ii} + x_i}{\sum_{j \in N} c_{ij} + x_i} D(N)_i, & i \in N, \\ D_{ij} = \dfrac{c_{ij}}{\sum_{j \in N} c_{ij} + x_i} D(N)_i, & j \in N \backslash i. \end{cases} \tag{1}$$

Indirect taxes are collected from the production sold within the region, by sales to other regions, and by imports from abroad (custom tariffs). Therefore:

$$\begin{cases} I_{ii} = \dfrac{c_{ii} + m_i}{\sum_{j \in N} c_{ij} + m_i} I_i, & i \in N, \\ I_{ij} = \dfrac{c_{ij}}{\sum_{j \in N} c_{ij} + m_i} I_i, & j \in N \backslash i. \end{cases} \tag{2}$$

The total taxes are $t_{ij} = D_{ij} + I_{ij}$. Hence, t_{ii} is the sum of (1) all *indirect* taxes (VAT and excise duties on alcoholic beverages, energy products and electricity, and manufactured tobacco) collected from the products of region i sold within the region, as well as tariff and import duties applied to imports from abroad; along with (2) *direct* taxes (from salaries, rents from capital, and corporate taxes) due to the production of goods sold within the region and exports abroad. Therefore, for each $j \in N \backslash i$, t_{ij} is the sum of all direct and indirect taxes originated from the production of goods and services in region i sold to region j. The total matrix is $T = D + I$.

Given a set of regions N and tax matrices $D, I \in \mathbb{R}_+^{N \times N}$, the pair $(N; D, I)$ is called a *tax problem*. The space of all tax problems with finite regions set N is denoted by $\mathcal{T}^{\mathrm{N}}$, and by $\mathcal{T}$ the space of all tax problems. A *tax allocation rule* ψ is a vector function $\psi: \mathcal{T}^{\mathrm{N}} \to \mathbb{R}^{\mathrm{N}}$, which for

[2] Vector and matrix notation: let $X = [x_{ij}]$ be a square matrix. For each $i \in N$, we denote the row i by $X_i = (x_{i1},\dots,x_{in})$, and the column i by $X^i = (x_{1i},\dots,x_{ni})$; the sum of its components by $X_i(N) = \sum_{j \in N} x_{ij}$, and $X^i(N) = \sum_{j \in N} x_{ji}$. The sum of all components of the matrix by $X(N) = \sum_{i \in N} \sum_{j \in N} x_{ij}$. And for all $S \subseteq N$, $X(S) = \sum_{i \in S} \sum_{j \in S} x_{ij}$.

each problem $(N; D, I)$ specifies how to redistribute all taxes collected among the regions. An obvious property that a tax allocation rule should satisfy is that it must share between regions the full tax budget at their disposition. This property is called *Efficiency*.

Definition 1 *We say that a tax rule ψ on $\mathcal{T}^N$ is efficient if $\sum_{i\in N}\psi_i(N; D, I) = T(N)$.*

A simple example of tax rule is the *population egalitarian* rule, denoted by Eg. This rule yields to each region i a share of the total budget in *proportion with its population*, in such a way that the expenses per capita are the same in every region. Let $w \in \mathbb{R}^N_{++}$ be a vector of population, where w_i is the population of region $i \in N$.

Definition 2 *For any tax problem $(N; D, I)$ the population egalitarian rule Eg is defined by*

$$Eg_i(N; D, I) = \frac{w_i}{w(N)} T(N), \qquad \forall i \in N.$$

The population egalitarian rule corresponds to the case of full solidarity between regions. Its purpose is that every citizen of the country should not be discriminated against with regard to public spending.

Given a tax allocation ψ, the *Fiscal Balances* are the differences between the total public spending allocated to each region and the total taxes collected within it.

Definition 3 *For any tax problem $(N; D, I)$ the fiscal balances FB associated to any tax rule ψ, is defined by $FB(\psi)_i(N; D, I) = \psi_i(N; D, I) - T_i(N)$, for all $i \in N$.*

Rich regions usually have negative fiscal balances, especially when there are wide differences between the per capita rent of the regions. If in addition, this also coincides with strong secessionist aspirations, these transfers of rent are presented as a sort of economic exploitation by central government. The FBs are presented as a measure of such grievances. Secession ideologists sometimes claim that their citizens would be richer if their region was an independent country, managing by themselves the total budget of T_i. However, fiscal balances yield a very naive measure of what happens under secession. In this breakdown scenario, changes appear in the inter-regional trade intensities (known as the 'border effect'), and modification in the collection of total direct and indirect taxes, due to the appearance of new borders with their corresponding custom tariffs. For this reason, we build the cooperative game v_t^α in order to give a more accurate idea of what a region, or a coalition of regions, could obtain under secession.

Suppose that a region, or more generally a coalition of regions $S \subset N$, break away from the country. In this secessionist scenario, new borders appear with economic consequences.

The first impact is known as the *border effect*. It establishes that domestic agents trade more with each other than with foreign agents of the same size and distance. This implies a fall in the trade intensity between regions in S and $N\backslash S$. We incorporate this fall in trade by a parameter α, $0 \leq \alpha \leq 1$. Now, for each $i \in S$ and $j \in N\backslash S$ the trade coefficients will be αc_{ij} and αc_{ji}. Hence, taxes collected will be affected in the same proportion. The parameter α reflects the optimistic/pessimistic perception of this breakdown process (the intensity of the border effect). The case $\alpha = 0$ corresponds to *the worst (most pessimistic) of possible breakdown scenarios*, in which all trade is interrupted. where $\alpha = 1$ is the most optimistic (very naive) one.

The second impact follows because part of intra-national trade converts into international trade between S and $N\backslash S$. Therefore, indirect taxes I_{ij} associated with trade from region $i \in S$ to region $j \in N\backslash S$ (collected initially in region i), now are collected as import tariffs in j in country $N\backslash S$. For the same reason, indirect taxes I_{ji} associated with the exports from j in the new country $N\backslash S$ to region i (collected initially in region j), now are collected in region i as import tariffs in country S. This change becomes particularly relevant when secessionist regions have a high positive net trade with the rest of the country.

We assume that within S and $N\backslash S$ the intensity of the intra-trade relationships does not change; hence they are able to manage $T(S)$ and $T(N\backslash S)$ respectively, irrespective the value of α.

Now we can define the tax game associated to the parameter α.

Definition 4 *Let a tax problem* $(N; D, I) \in \mathcal{T}^N$ *and coalition* $S \subseteq N$. *Given a parameter* α, *with* $0 \leq \alpha \leq 1$, *we define the tax game* v_t^α *by*

$$v_t^\alpha(S) = T(S) + \alpha \sum_{i \in S} \sum_{j \in N\backslash S} \left(D_{ij} + I_{ji}\right), \qquad \forall S \subseteq N. \qquad (3)$$

Note that by construction, for all α it holds that $v_t^\alpha(N) = T(N)$, and $v_t^\alpha(S) \leq v_t^{\alpha'}(S)$ for all $\alpha < \alpha'$ and $S \subseteq N$.

Now we are ready to analyze the stability of a particular tax allocation $x \in \mathbb{R}_+^N$. This is done by computing the difference between what the allocation gives to every group of regions $S \subseteq N$ and what the group could obtain by themselves if they were to withdraw from the country.

We recall some basic definitions of cooperative games in transferable utility form. A *transferable utility game* (TU-game for short) is given by a pair (N, v) where N is a finite set of players, $|N| = n$, and $v: 2^N \to \mathbb{R}$ is a *characteristic function*, which assigns to every coalition $S \subseteq N$ a real number $v(S)$, satisfying $v(\emptyset) = 0$. 2^N denote the set of all subsets of N, and for all $S \subseteq N$, $v(S)$ is called the *worth* of S. We denote by (N, v) the *transferable utility game* with player set N and *characteristic function* v. The space of all games with finite player set N is denoted by $\mathcal{G}^N$, and by $\mathcal{G}$ the space of all games. Given a game (N, v) and coalition S, we write (S, v) for the subgame obtained by restricting v to subsets of S only (i.e., to 2^S).
A payoff allocation is any vector $x = (x_1, \ldots, x_n) \in \mathbb{R}^N$, where each component x_i is the utility payoff of player i. Given an allocation x we use the notation $x(S) = \sum_{i \in S} x_i$ for every $S \subseteq N$. A *solution* is a function ψ which assigns a real number $\psi_i(N, v)$ to every game (N, v) and every player $i \in N$.
From the data of any *tax problem* $(N; D, I) \in \mathcal{T}^N$ and parameter $\alpha \in [0,1]$, we associate a *cooperative tax game* v_t^α as given in (1). This constitutes total taxes that the coalition of regions S can collect. We will say that $(N, v_t^\alpha) \in \mathcal{G}^N$ is the *tax game* associated to the *tax problem* $(N; D, I) \in \mathcal{T}^N$, for every parameter α.
An obvious property that a value ψ should satisfy is that it must share the full worth at their disposition among players. This property is called *Efficiency*. The definition in the context of cooperative games $\mathcal{G}^N$ is the same as was made for tax problems $\mathcal{T}^N$.

Definition 5 *We say that a solution* ψ *on* $\mathcal{G}^N$ *is efficient if* $\sum_{i \in N} \psi_i(N, v) = v(N)$.

We say that an allocation x is *feasible for a coalition* S if $x(S) \leq v(S)$. So the players in S can achieve their components of x by dividing among themselves the worth of $v(S)$ that they can get if they cooperate together. The *excess of* S *in* x, denoted by $e(S, x, v)$, is defined by $e(S, x, v) = v(S) - x(S)$. The excess can be seen as a measure of the *dissatisfaction* of coalition S with the allocation x.
We say that a coalition S can *improve* an allocation x if there exists some allocation y such that y is feasible for S, i.e. $y(S) \leq v(S)$, and all players i in S get $y_i > x_i$. It is clear that in such a

case $e(S,x,v)>0$. An allocation x is in the *core* of v, called $C(N,v)$, if x is efficient and no coalition can improve on x (Gillies, 1953). That is,

$$C(N,v)=\{x\in\mathbb{R}^N|\ \ x(N)=v(N)\quad\wedge\quad e(S,x,v)\leq 0,\ \forall S\subseteq N, S\neq\emptyset\}.$$

Core notions had played important roles in Economic Theory, as the core of a Walrasian economy, or the set of stable matchings. In our tax federalism setting, the core of the tax game will be the set of stable tax allocations.

Definition 6 *Given a tax problem* $(N;D,I)$ *and parameter* α *we say that a tax allocation* x *is stable if* $x\in C(N,v_t^\alpha)$.

The case $\alpha=0$ corresponds to the most pessimistic perception. In such a case, payoffs x *outside the core* of the game, are clearly *unstable*. This implies that there exists a coalition of regions S which could improve the payoffs of x by cooperating together and dividing total taxes $v_t^0(S)$ among themselves, *even in the worst (most pessimistic) of possible breakdown scenarios*, where all trade relationships are interrupted and all its associated taxes are lost.
However, for payoffs $x\in C(N,v_t^0)$ we can still have doubts whether they can qualify as stable. This will depend of our perception of what is the *true* value for α. As, by construction, it holds that $v_t^\alpha(S)\leq v_t^{\alpha'}(S)$ for all $\alpha\leq\alpha'$, it follows that $C(N,v_t^{\alpha'})\subseteq C(N,v_t^\alpha)$. If the optimistic perception about the economic possibilities of a coalition in case of secession increases, the set of stable payoffs is reduced. In the optimistic case of $\alpha=1$, the tax game v_t^1 is a totally additive characteristic function[3]. This implies that the core reduces to a single point: $C(N,v_t^1)=\left\{\left(v_t^1(i)\right)_{i\in N}\right\}$.
What about the existence of stable allocations? Calvo (2018) shows that the game v_t^α is convex, which implies that its core is non-empty. The basic result there is the following:

Theorem 7 *The core* $C(N,v_t^\alpha)$ *is non-empty for every* $0\leq\alpha\leq 1$.

In Calvo (2018) we show that neither the present regional tax financial system nor the population egalitarian tax rules satisfy stability for the Spanish case, even for the most pessimistic case of $\alpha=0$.

3 TAX RULES

The purpose of this section is to present some rules which satisfy stability. It is clear that for the most optimistic case $\alpha=1$, there is no freedom to choose, as there is a unique stable tax allocation. However, $\alpha=1$ is an unrealistic economic scenario, and this opens the door to searching for tax rules that satisfy some degree of stability.

3.1 Balanced tax rule

The purpose of our first tax rule, is to balance the contribution of welfare that each region makes to each other in form of tax transfers. This idea embodies a kind of reciprocity principle: if a region j contributes to the welfare of i more than the contribution of i to j, then i is in debt to j, and i should compensate j with a monetary transfer.

We start to measure such wealth contributions. So, let $C=\left[c_{ij}\right]$ be the data of commercial relationship within a country. For each pair of regions $i,j\in N$, c_{ij} is the amount of goods and services of region i sold to region j. Hence, c_{ij} measures in monetary units the contribution of region j to the welfare of region i. Accordingly, c_{ji} measures the contribution of region i to the

[3] That is, $v_t^1(S)=\sum_{i\in S}v_t^1(i)$ for all $S\subseteq N$.

welfare of j. As the taxes collected are in proportion to the economic activities measured by these coefficients, the amount t_{ij} is the sum of all direct and indirect taxes that region i loses from the total taxes collected, if region j withdraws from the tax system and *stops buying all products and services* from i. That is

$$t_{ij} = T_i(N) - T_i(N \backslash j)$$

As all regions participate in the same common tax system, we can measure these impacts in the aggregate. That is, let $\sum_{j \in N \backslash i} t_{ij}$ be the total contribution of the remaining regions to the taxes collected by region i, and $\sum_{j \in N \backslash i} t_{ji}$ be the total contribution of region i to the taxes collected by the remaining regions. Now, when $\sum_{j \in N \backslash i} t_{ij} > \sum_{j \in N \backslash i} t_{ji}$, the remaining regions contribute more to the taxes of region i than i contributes to the taxes of the rest. In this case, region i is in debt with all other regions and it should be "fair" that i compensates the system with some monetary transfer. As T_i is the sum of all taxes collected in region i, and $\psi_i(N; D, I)$ is the total public spending in region i, determined by the tax rule ψ, this region makes a positive transfer if $T_i - \psi_i(N; D, I) > 0$.

We can enunciate this *reciprocity* principle as follows.

Definition 8 *We say that ψ satisfies reciprocity if for any tax problem $(N; D, I)$ it holds that $\left(\sum_{j \in N \backslash i} t_{ij} - \sum_{j \in N \backslash i} t_{ji}\right) \cdot \left(T_i(N) - \psi_i(N, T)\right) > 0$, for each $i \in N$ whenever $\sum_{j \in N \backslash i} t_{ij} \neq \sum_{j \in N \backslash i} t_{ji}$.*

Note that $\sum_{j \in N \backslash i} t_{ij} - \sum_{j \in N \backslash i} t_{ji} = T_i(N) - T^i(N)$. Therefore, if a rule ψ satisfies reciprocity, the differences, $T_i(N) - T^i(N)$ and $T_i(N) - \psi_i$, must have both the same sign.

There is a simple example of a tax rule which doesn't satisfy reciprocity. We can define it as the *no transfers rule*, denoted by ξ.

Definition 9 *For any tax problem $(N; D, I)$, the no transfers rule ξ is defined by $\xi_i(N; D, I) = T_i(N)$, for all $i \in N$.*

Now each region has at its disposal all taxes it has collected itself. Here, there is no tax transfer at all among regions. Therefore, if $T_i(N) \neq T^i(N)$ then we have no reciprocity, as $\xi_i = T_i(N)$ by definition.

On the other side, it is easy to find tax rules which satisfy this property. Take for example, the family ξ^λ of tax rules defined by

$$\xi_i^\lambda(N; D, I) = \lambda T_i(N) + (1 - \lambda) T^i(N), \ \forall i \in N, \ \forall \lambda \in [0,1].$$

Proposition 10 *ξ^λ satisfies reciprocity for all $0 \leq \lambda < 1$.*

Proof For all $i \in N$, we have that $T_i(N) - \xi_i^\lambda(N; D, I) = (1 - \lambda)\left(T_i(N) - T^i(N)\right)$, then $T_i(N) - \xi_i^\lambda(N; D, I) > 0$ if and only if $T_i(N) - T^i(N) > 0$ ■

As reciprocity is an *ordinal* property, it allows for the existence of many rules satisfying this property. For this reason, we introduce a *cardinal* property which specifies how much the payoffs of a region should vary when another region drops out of the system.
Given a tax problem $(N; D, I)$ and region $i \in N$, we define $(N \backslash i; D^{-i}, I^{-i})$ as the sub-problem which arises when region i withdraws from set N (breaking all commercial relationships with

the rest). The new tax matrices D^{-i}, I^{-i} are obtained by deleting the row and column i from the initial tax matrices D, I. This is consistent with the definition of the tax game v_t^0, because $v_t^0(N\backslash i) = T^{-i}(N\backslash i)$. Therefore, the subgame associated with the sub-tax problem $(N\backslash i; D^{-i}, I^{-i})$ is just $(N\backslash i, v_t^0)$. Now, given an allocation rule ψ defined in a tax problem $(N; D, I)$, we measure the payoff variation obtained by j when i leaves the tax system: $\psi_j(N; D, I) - \psi_j\big((N\backslash i; D^{-i}, I^{-i}\big)$; and the payoff variation of i when j leaves: $\psi_i(N; D, I) - \psi_i\big((N\backslash j; D^{-j}, I^{-j}\big)$. We say that two regions are in *balance* when such variation in payoffs are the same for both. We may say equivalently that the budget contribution of region i to j is the same as the budget contribution of j to i, for each pair of regions $\{i, j\} \subseteq N$.

Definition 11 *We say that ψ is balanced if $\psi_i(N; D, I) - \psi_i\big(N\backslash j; D^{-j}, I^{-j}\big) = \psi_j(N; D, I) - \psi_j\big(N\backslash i; D^{-i}, I^{-i}\big)$ for every pair of regions $\{i, j\} \subseteq N, i \neq j$, and every tax problem $(N; D, I)$.*

Alternatively, the *maximum damage* that i could inflict to the tax budget received by j is by withdrawing from the tax system and breaking all trade relationships with the remaining regions. In such a case, the variation in the tax budget of j is equal to $\psi_j(N; D, I) - \psi_j\big(N\backslash i; D^{-i}, I^{-i}\big)$. If the tax rule satisfies this property, the damage that every region could inflict on each other are in balance. Obviously, this maximum damage corresponds to the most pessimistic scenario in which all trade relations are lost when a region leaves the tax system.

The first question to solve is the *existence* and *uniqueness* of an efficient and balanced tax rule, which will be called a *balanced tax rule* and denoted by φ. This is the content of the next result.

Theorem 12 *There exists a unique tax rule on $\mathcal{T}$ which is efficient and balanced. This is the balanced tax rule φ.*

Proof We use induction in the cardinality of N.
Let $N = \{i\}$. Therefore $D = [D_{ii}]$ and $I = [I_{ii}]$. By efficiency, $\varphi_i(\{i\}; [D_{ii}], [I_{ii}]) = t_{ii}$, and then φ is determined uniquely.
Let now $(N; D, I)$ be a tax problem such that $|N| \geq 2$.
Assuming that φ is balanced, this property implies that

$$\varphi_i(N; D, I) = \varphi_j(N; D, I) - \varphi_j\big(N\backslash i; D^{-i}, I^{-i}\big) + \varphi_i\big(N\backslash j; D^{-j}, I^{-j}\big), \qquad \forall j \in N\backslash i.$$

Therefore, adding up over all $j \in N\backslash i$ and summing the term $\varphi_i(N, T)$ on both sides of the equation we have that

$$n \cdot \varphi_i(N; D, I) = \sum_{j \in N} \varphi_j(N; D, I) - \sum_{j \in N\backslash i} \varphi_j\big(N\backslash i; D^{-i}, I^{-i}\big) + \sum_{j \in N\backslash i} \varphi_i\big(N\backslash j; D^{-j}, I^{-j}\big),$$

and applying efficiency, it holds that

$$\varphi_i(N; D, I) = \frac{1}{n}(T(N) - T(N\backslash i)) + \sum_{j \in N\backslash i} \varphi_i\big(N\backslash j; D^{-j}, I^{-j}\big). \qquad (4)$$

By induction, we assume that every $\varphi_i\big(N\backslash j; D^{-j}, I^{-j}\big)$ exists and is determined uniquely. This implies that $\varphi_i(N; D, I)$ also exists and is determined uniquely.

Moreover, it is easy to check that φ, defined as in (4), also satisfies efficiency. ■

Recall that $T(N) - T(N\backslash i) = t_{ii} + \sum_{j \in N\backslash i}(t_{ij} + t_{ji})$, therefore equation (4) yields a *recursive* formula to compute φ, starting by $\varphi_i(\{i\}; [D_{ii}], [I_{ii}]) = t_{ii}$, for all $i \in N$.

Next we offer a direct formula to compute φ.

Definition 13 *For any tax problem* $(N; D, I) \in \mathcal{T}^N$, *the tax rule* φ *is defined by*

$$\varphi_i(N; D, IT) = t_{ii} + \frac{1}{2}\sum_{j \in N\backslash i}(t_{ij} + t_{ji}), \qquad \forall i \in N. \tag{5}$$

The easiest way to see that formulas (4) and (5) are equivalent is to check that the rule defined by (5) is efficient and balanced. As there is a unique rule which satisfies both properties, (4) and (5) must coincide.

Proposition 14 *The tax rule* φ *defined by* (5) *is efficient and balanced.*

Proof That (5) implies efficiency is straightforward. To see that (5) implies also balanceness, take any pair $i, j \in N$, then

$$\varphi_i(N; D, I) - \varphi_i(N\backslash j; D^{-j}, I^{-j}) =$$

$$= \left(t_{ii} + \frac{1}{2}\sum_{k \in N\backslash i}(t_{ik} + t_{ki})\right) - \left(t_{ii} + \frac{1}{2}\sum_{k \in N\backslash\{i,j\}}(t_{ik} + t_{ki})\right)$$

$$\tfrac{1}{2}(t_{ij} + t_{ji}) = \varphi_j(N; D, I) - \varphi_j(N\backslash i; D^{-i}, I^{-i}). \quad \blacksquare$$

Notice that if a tax rule ψ satisfies this bilateral balanced property, it holds automatically that the rule is also *balanced in the aggregate*, because adding up over all $j \in N\backslash i$, it holds that

$$\sum_{j \in N\backslash i}\left(\boldsymbol{\varphi_i(N; D, I) - \varphi_i(N\backslash j; D^{-j}, I^{-j})}\right) = \sum_{j \in N\backslash i}\left(\boldsymbol{\varphi_j(N; D, I) - \varphi_j(N\backslash i; D^{-i}, I^{-i})}\right). \tag{6}$$

That is, the sum of the budget contributions of the remaining regions to region i is equal to the sum of the budget contributions of i to all other regions[4].
Formula (5) also shows that φ is individually rational: $\varphi_i(N; D, I) \geq t_{ii} = v_t^0(i)$, for all $i \in N$. The tax rule φ gives to each region i firstly, all direct and indirect taxes t_{ii} collected within region, as a result of its internal commercial activity c_{ii}, its exports x_i, and its imports from abroad m_i. Secondly, the additional amount of taxes collected due to the relationship between regions i and j originated by their bilateral trade, $t_{ii} + t_{ji}$, is equally distributed between them. Now we see that φ satisfies reciprocity.

Proposition 15 φ *satisfies reciprocity on* $\mathcal{T}^N$.

Proof From (5) we see that $\varphi_i(N; D, I) = \frac{1}{2}\left(T_i(N) + T^i(N)\right) = \xi^{\frac{1}{2}}(N; D, I)$, for all $i \in N$.

[4] It is immediate that efficiency and balanced in the aggregate also characterizes φ in $\mathcal{T}^N$. This follows as efficiency and (¡Error! solo el documento principal.) imply the recursive formula (¡Error! solo el documento principal.).

Then reciprocity is satisfied by Proposition (14) ■

Independently of how appealing the properties of efficiency and balance might be, as they offer an axiomatic support of φ (or the properties of balanced in the aggregate and reciprocity); our main target is to find a stable tax rule.

We notice that φ is built in such a way that we equalize the *maximum* damage that every region inflicts on each other by withdrawing the system. That is, assuming that the most pessimistic scenario occurs: all trade relationships are broken. In the next theorem we show that φ belongs to $C(N, v_t^0)$ for every tax problem. This is the main stability result of this section. However, it can be argued that the characteristic function v_t^0 is too *pessimistic* about how much tax a coalition of regions S can obtain when they withdraw from the system. It can be objected to that this extreme assumption is unrealistic, and we can expect that some trade activity would remain after the breakdown to a greater or less extent. As far as α increases, the set of stable tax allocations $C(N, v_t^\alpha)$ reduces. Therefore, the balanced tax rule φ will be not stable from some values of α. And this threshold value will depend on the data of the tax problem. Fortunately, we can enlarge the stability result: it is possible to find a minimal bound of α which guarantees that φ is still stable in every tax problem. This is the content of the next result.

Theorem 16 *The balanced tax rule φ is stable on $\mathcal{T}^N$ for every $\alpha \leq \frac{1}{2}$.*

Proof Let $(N; D, I) \in \mathcal{T}^N$. Firstly, φ is efficient by Proposition 14. Secondly, let a coalition $S \subseteq N$, therefore

$$
\begin{aligned}
e(S, \varphi, v_t^\alpha) = v_t^\alpha(S) - \varphi(N; D, I)(S) \\
&= T(S) + \alpha \sum_{i \in S} \sum_{j \in N \setminus S} (D_{ij} + I_{ji}) - \left(T(S) + \frac{1}{2} \sum_{i \in S} \sum_{j \in N \setminus S} (t_{ij} + t_{ji}) \right) \\
&= \left(\alpha - \frac{1}{2} \right) \sum_{i \in S} \sum_{j \in N \setminus S} (D_{ij} + I_{ji}) - \frac{1}{2} \sum_{i \in S} \sum_{j \in N \setminus S} (I_{ij} + D_{ji}).
\end{aligned}
$$

Then, if $\alpha \leq \frac{1}{2}$ it holds that $e(S, \varphi, v_t^\alpha) \leq 0$, and then $\varphi(N; D, I) \in C(N, v_t^\alpha)$. ■

3.2 Redistributive balanced tax rule

In the discussion of the design of a regional financial system, solidarity among regions usually appears recurrently. For example, this idea is expressed in De la Fuente *et al.* (2016) by the following principle:

- *Non-discrimination*: Distributed funds must provide a uniform level of public services throughout the country.

Hence, solidarity will imply transfers of rent from rich to poor regions, when wide differences exist in their per capita rent. Therefore, redistribution and stability can be incompatible in some countries. The Eg tax rule is the most egalitarian, because it equalizes the per capita budget spent in each region. We show in Calvo (2018) that Eg yields an unstable tax allocation[5] in Spain. The balanced tax rule φ designed in the previous section is the most stable one, because it is

[5] Given the differences that also exist in the per capita rent between the northern and southern regions of Italy, we can conjecture that the Eg tax rule would also be unstable in Italy.

placed in the center of the core of the pessimistic tax game. This fact will be shown in Section 5. As redistribution was not an initial purpose when designing the rule φ, we cannot expect good behavior in such a direction when it is applied in countries with nonhomogeneous rent distribution.

The purpose of this section is to design a tax rule with some degree of rent redistribution. We try to follow one of the principles also expressed in De la Fuente *et al.* (2016):

- *Fairness in redistribution*: Allocated funds should vary *directly* depending on fiscal needs and *inversely* with the tax capacity of each jurisdiction.

A way to accomplish this principle is to take into account the differences in the per capita rent of regions when they share the tax benefits between them. Given a pair of regions $i, j \in N$, we can obtain a redistributive effect, if their tax benefits $(t_{ij} + t_{ji})$ are shared inversely proportional to their per capita wealth level.

Let $(N; D, I)$ be a tax problem, $w = (w_i)_{i\in N} \in \mathbb{R}^N_{++}$ the vector of population, and $IGDP = (IGDP_i)_{i\in N} \in \mathbb{R}^N_{++}$ the vector of initial gross domestic product[6]. And let p_i be the initial gross domestic product per capita of each region $i \in N$, i.e. $p_i = \frac{IGDP_i}{w_i}$.

Definition 17 *For any tax problem $(N; D, I) \in \mathcal{T}^N$, the p-balanced tax rule φ^p is defined by*

$$\boldsymbol{\varphi_i^p(N; D, I) = t_{ii} + \sum_{j\in N\setminus i} \frac{p_j}{p_i + p_j}\left(t_{ij} + t_{ji}\right),}$$
$$\forall \boldsymbol{i} \in \boldsymbol{N}. \qquad (7)$$

We see now that φ^p satisfies a weighted version of the balanced property.

Definition 18 *Given a vector of weights $p \in \mathbb{R}^N_{++}$. We say that ψ is a p-balanced tax rule if* $p_i\left(\psi_i(N; D, I) - \psi_i\left(N\setminus j; D^{-j}, I^{-j}\right)\right) = p_j\left(\psi_j(N; D, I) - \psi_j\left(N\setminus i; D^{-i}, I^{-i}\right)\right)$ *for every pair of regions $\{i, j\} \subseteq N, i \neq j$, and every tax problem $(N; D, I)$.*

This property says that two regions are in *redistributive balance* when the contribution of region i to the budget variation of region j, $\psi_j(N; D, I) - \psi_j\left(N\setminus i; D^{-i}, I^{-i}\right)$, is *directly proportional* to its per capita richness level p_i:

$$\frac{p_i}{p_j} = \frac{\psi_j(N; D, I) - \psi_j\left(N\setminus i; D^{-i}, I^{-i}\right)}{\psi_i(N; D, I) - \psi_i\left(N\setminus j; D^{-j}, I^{-j}\right)}$$

Proposition 19 *The tax rule φ^p is efficient and p-balanced.*

Proof Check efficiency is straightforward. To see that φ^p is a p-balanced tax rule, note that, by (7), $\varphi_i^p(N; D, I) - \varphi_i^p\left(N\setminus j; D^{-j}, I^{-j}\right) = \frac{p_j}{p_i+p_j}\left(t_{ij} + t_{ji}\right)$,
and then it holds that

[6] *Initial* means before any redistribution of rent among regions.

$$p_i\left(\varphi_i^p(N;D,I)-\varphi_i^p\left(N\backslash j;D^{-j},I^{-j}\right)\right)=\frac{p_ip_j}{p_i+p_j}\left(t_{ij}+t_{ji}\right)$$
$$=p_j\left(\varphi_j^p(N;D,I)-\varphi_j^p\left(N\backslash i;D^{-i},I^{-i}\right)\right). \quad \blacksquare$$

Theorem 20 *Given a vector of weights $p\in\mathbb{R}_{++}^N$. There exists a unique tax rule on $\mathcal{T}$ which is efficient and p-balanced. This is the p-balanced tax rule φ^p.*

Proof We use induction in the cardinality of N.
Let $N=\{i\}$. Therefore $D=[D_{ii}]$ and $I=[I_{ii}]$. By efficiency, $\varphi_i^p(\{i\};[D_{ii}],[I_{ii}])=t_{ii}$, and then φ^p is determined uniquely.
Let now $(N;D,I)$ be a tax problem such that $|N|\geq 2$.
Assuming that φ^p is p-balanced, this implies that

$$\frac{1}{p_j}\varphi_i^p(N;D,I)=\frac{1}{p_i}\varphi_j^p(N;D,I)-\frac{1}{p_i}\varphi_j^p\left(N\backslash i;D^{-i},I^{-i}\right)+\frac{1}{p_j}\varphi_i^p\left(N\backslash j;D^{-j},I^{-j}\right),\qquad \forall j\in N\backslash i.$$

Therefore, adding up over all $j\in N\backslash i$ and summing the term $\frac{1}{p_j}\varphi_i^p(N;D,I)$ in both sides of the equation we have that

$$\left(\sum_{j\in N}\frac{1}{p_j}\right)\varphi_i^p(N;D,I)=$$
$$=\frac{1}{p_i}\sum_{j\in N}\varphi_j^p(N;D,I)-\frac{1}{p_i}\sum_{j\in N\backslash i}\varphi_j^p\left(N\backslash i;D^{-i},I^{-i}\right)+\sum_{j\in N\backslash i}\frac{1}{p_j}\varphi_i\left(N\backslash j;D^{-j},I^{-j}\right).$$

and applying efficiency, it holds that

$$\varphi_i^p(N;D,I)=\frac{\frac{1}{p_i}}{\left(\sum_{j\in N}\frac{1}{p_j}\right)}\cdot\left(T(N)-T(N\backslash i)\right)+\sum\nolimits_{j\in N\backslash i}\frac{\frac{1}{p_j}}{\left(\sum_{j\in N}\frac{1}{p_j}\right)}\varphi_i\left(N\backslash j;D^{-j},I^{-j}\right). \tag{8}$$

By induction, we assume that every $\varphi_i^p\left(N\backslash j;D^{-j},I^{-j}\right)$ exists and is determined uniquely. This implies that $\varphi_i^p(N;D,I)$ also exists and is determined uniquely.
Moreover, it is easy to check that φ^p defined by (8) also satisfies efficiency. ■

Notice that formula (8) provides a *recursive* way to compute (7).

Now φ^p is not in the center of the core in the pessimistic tax game. For that reason, we can only guarantee stability in the most pessimistic case. That is, no matter how the per capita wealth level of regions differs, the redistributive effect of the tax rule always yields a stable outcome when $\alpha=0$.

Theorem 21 *The redistributive balanced tax rule φ^p is stable on $\mathcal{T}^N$ for all $p\in\mathbb{R}_{++}^N$ and $\alpha=0$.*

Proof Let $(N;D,I)\in\mathcal{T}^N$. Firstly, φ^p is efficient by Proposition 19. Secondly, let a coalition $S\subseteq N$, therefore

$$e(S, \varphi^p, v_t^0) = v_t^0(S) - \varphi^p(N; D, I)(S) = T(S) - \left(T(S) + \sum_{i \in S} \sum_{j \in N \setminus S} \frac{p_j}{p_i + p_j} (t_{ij} + t_{ji}) \right)$$

$$= - \sum_{i \in S} \sum_{j \in N \setminus S} \frac{p_j}{p_i + p_j} (t_{ij} + t_{ji}) \le 0. \quad \blacksquare$$

3.3 Optimistic secession rule

We can compute also how much tax a region can expect to collect in case of secession, for every value of α.

Definition 22 *For any tax problem $(N; D, I) \in \mathcal{T}^N$ and parameter α, $0 \le \alpha \le 1$, the secessionist tax rule ζ^α is defined by*

$$\zeta_i^\alpha(N; D, I) = t_{ii} + \alpha \sum_{j \in N \setminus i} (D_{ij} + I_{ji}), \qquad \forall i \in N.$$

When $\alpha = 0$, every region can only expect to obtain its individually rational payoffs, i.e. $\zeta_i^0(N; D, I) = t_{ii}$. The value ζ_i^α computes the individually rational payoffs of the tax game v_t^α; that is, $\zeta_i^\alpha(N; D, I) = v_t^\alpha(i)$. Notice that, for values $\alpha < 1$, ζ^α is not an efficient allocation, because generally

$$v_t^\alpha(N) - \zeta^\alpha(N) = (1 - \alpha) \sum_{i \in S} \sum_{j \in N \setminus S} (D_{ji} + I_{ij}) \ne 0.$$

Only when $\alpha = 1$ does efficiency hold. In this optimistic case we call ζ^1 the *optimistic secession rule*. We show now that ζ^1 is always stable.

Proposition 23 *Let $(N; D, I) \in \mathcal{T}^N$ be a tax problem. Then it holds that $\zeta^1(N; D, I) \in C(N, v_t^\alpha)$, for all α, $0 \le \alpha \le 1$.*

Proof Let a coalition $S \subset N$ and $0 \le \alpha \le 1$, then we have that

$$\zeta^1(N; D, I)(S) = T(S) + \sum_{i \in S} \sum_{j \in N \setminus S} (D_{ji} + I_{ij}) \ge T(S) + \alpha \sum_{i \in S} \sum_{j \in N \setminus S} (D_{ji} + I_{ij}) = v_t^\alpha(S)$$

hence $y \in C(N, v_t^\alpha)$. ■

Moreover, it is straightforward to check that when $\alpha = 1$, the allocation $\zeta^1(N; D, I)$ is the *unique* stable allocation in the game (N, v_t^1). This happens because v_t^1 is an additive characteristic function, and then $C(N, v_t^1) = \{\zeta^1(N; D, I)\}$. It is clear that under a full optimistic scenario $\alpha = 1$, a secessionist supporter will only accept tax payoffs no less than the threshold marked by ζ^1. However, such extreme assumption contradicts the empirical evidence regarding the economic impact of the border effect. Anyway, we also have a parallel result for theorem 23, showing that $\zeta^{1/2}$ is a lower bound of φ.

Theorem 24 *For any tax problem $(N; D, I) \in \mathcal{T}^N$ it holds that $\varphi_i(N; D, I) \ge \zeta_i^{1/2}(N; D, I)$ for all $i \in N$.*

Proof Let $(N; D, I) \in \mathcal{T}^N$ be a tax problem and $S \subseteq N$. Then, for all $i \in N$ we have that

$$\varphi_i(N; D, I) = t_{ii} + \frac{1}{2} \sum_{j \in N \setminus i} \left(D_{ij} + I_{ij} + D_{ji} + I_{ji} \right) \geq$$

$$t_{ii} + \frac{1}{2} \sum_{j \in N \setminus i} \left(D_{ij} + I_{ji} \right) = \zeta_i^{1/2}(N; D, I). \quad \blacksquare$$

3.4 A numerical example

We use a simple numerical example to illustrate the concepts defined up to now.

Example. Let $N = \{1,2,3\}$ be a country with three regions, and with the following tax matrices:

$$D = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 1 \\ 0 & 2 & 3 \end{bmatrix}; \quad I = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 2 & 2 \\ 1 & 1 & 2 \end{bmatrix}; \quad T = D + I = \begin{bmatrix} 2 & 0 & 1 \\ 2 & 4 & 3 \\ 1 & 3 & 5 \end{bmatrix}.$$

The coefficient $t_{12} = 0$ comes from the fact that region 1 does not export goods and services to region 2. Therefore, there are no direct taxes, nor indirect taxes associated with the trade from region 1 to region 2.

The associated tax game (N, v_t^α) is given by

$v_t^\alpha(1) = 2, v_t^\alpha(2) = 4, v_t^\alpha(3) = 5,$

$v_t^\alpha(\{1,2\}) = 8 + \alpha(1+2), v_t^\alpha(\{1,3\}) = 9 + \alpha(2+3), v_t^\alpha(\{2,3\}) = 15 + \alpha(1+1),$

$v_t^\alpha(\{1,2,3\}) = 21.$

To see the inequalities that satisfy any stable allocation x, note that, for region 1 it must hold that $x_1 \geq v_t^\alpha(1) = 2$, and that $x_2 + x_3 \geq v_t^\alpha(\{2,3\}) = 15 + 2\alpha$, which jointly with $x(N) = v_t^\alpha(\{1,2,3\}) = 21$, implies that $x_1 \leq x(N) - v_t(N) = 21 - 15 - 2\alpha = 6 - 2\alpha$. Following the same reasoning for regions 2 and 3, we realize that the core of the game (N, v_t^α) is given by the set

$$C(N, v_t^\alpha) = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : 2 \leq x_1 \leq 6 - 2\alpha; 4 \leq x_2 \leq 12 - 5\alpha; 5 \leq x_3 \leq 13 - 3\alpha\}.$$

We can check that the payoffs of the no transfers rule ξ are

$$\xi_1(N, T) = T_1(N) = 3, \xi_2(N, T) = T_2(N) = 9, \xi_3(N, T) = T_3(N) = 9.$$

For example, for $\alpha = 0$, the set of stable allocations is the convex hull of the following vertices:

$$C(N, v_t^0) = C_H\{(6,4,11), (4,4,13), (2,6,13), (2,12,7), (4,12,5), (6,10,5)\}$$

We can represent graphically all these points with the help of the equilateral triangle given in Figure 1. This triangle represents all efficient and non-negative division payoffs. For example, for region 3, the bottom side represents allocations where region 3 obtains zero. The vertex (0,0,21) is the opposite case, where region 3 obtains the total taxes to share, i.e. 21. Each intermediate horizontal line represents a constant payoff for region 3 (between 0 and 21). The same happens for regions 1 and 2, with parallel lines of constant payoffs, between the maximum payoff of 21 in the vertex and the minimum 0 payoff in its opposite side.

The core $C(N, v_t^0)$ of this game is the shadow area of the hexagon in Figure 1, and we can see that the no transfer rule $\xi(N,T) = (3,9,9)$ is placed inside the core.

Now assume that the population of the three regions is given by the vector $w = (12,4,5)$. The payoffs of the population egalitarian rule are

$$Eg_1(N,T) = \frac{12}{21} \cdot 21 = 12, Eg_1(N,T) = \frac{4}{21} \cdot 21 = 4, Eg_1(N,T) = \frac{5}{21} \cdot 21 = 5.$$

This point $(3,9,9)$ is not in the core, because

$$e(\{2,3\}, Eg, v_t^0) = v_t^0(\{2,3\}) - (Eg_2 + Eg_3) = 15 - (4+5) = 6 > 0.$$

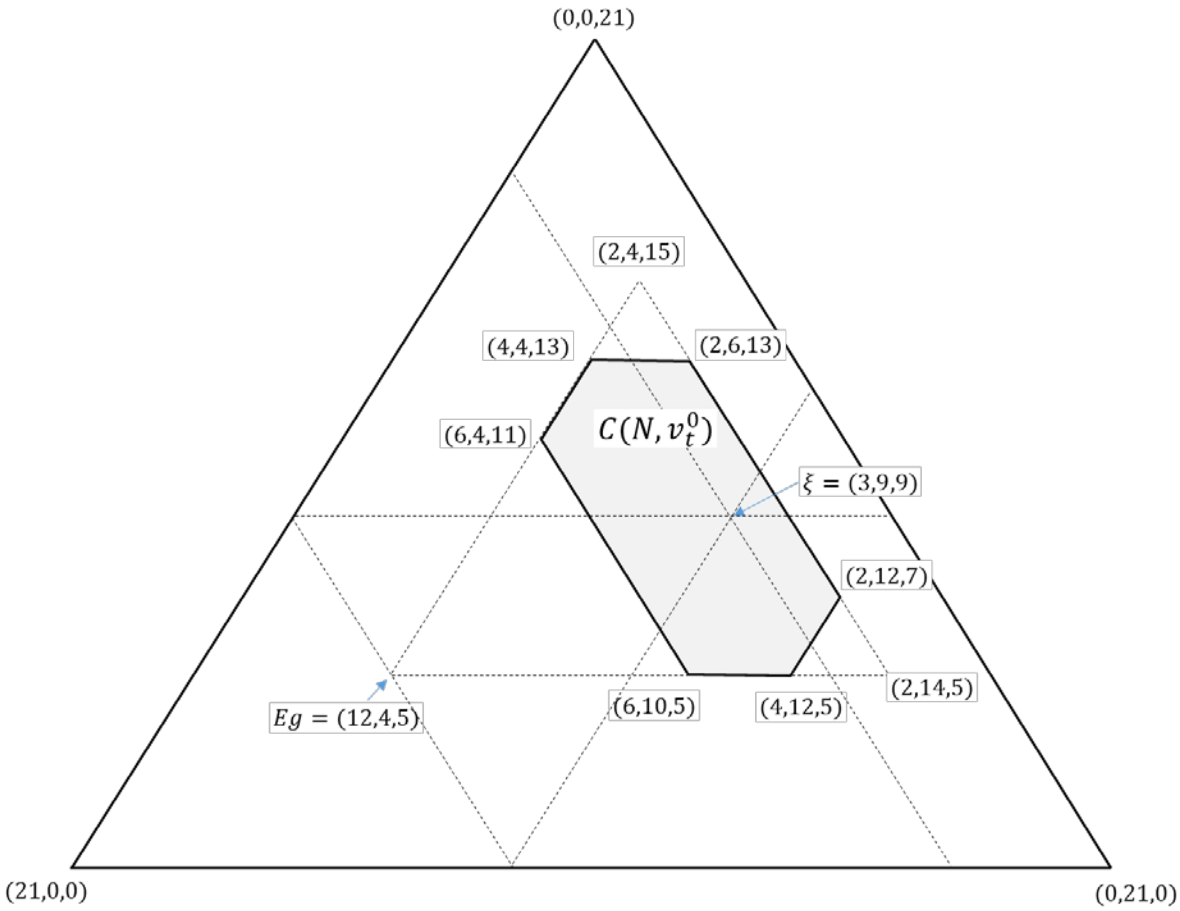

Figure 1. Efficient allocations

Now, assume that the vector of initial gross domestic product is $IGDP = (12, 30, 30)$. Therefore, the vector of initial gross domestic product per capita is $p = (1, 7.5, 6)$. There is a wide difference between the per capita rent of Region 1 and Regions 2 and 3. The balanced tax rule yields as payoffs $\varphi = (4, 8, 9)$, and its associated final gross domestic product $\varphi GDP_i = IGDP_i + (\varphi_i - T_i)$, for $i = 1,2,3$. That is, $\varphi GDP = (13, 29, 30)$, and the final gross domestic product per capita associated with φ, $p' = (1.083, 7.25, 6)$. It is clear that with φ we obtain a poor effect in the wealth redistribution comparing p with p'. If we compute now the redistributive tax rule we obtain the associated vectors of payoffs $\varphi^p = (5.479, 6.902, 8.619)$, $\varphi^p GDP = (14.479, 27.902, 29.619)$, and the final gross domestic product per capita associated to φ^p, $p'' = (1.207, 6.975, 5.924)$. With φ^p we obtain a greater redistributive effect than φ, remaining still in the area of pessimistic stable payoffs. Obviously, with the population egalitarian rule Eg we can obtain a bigger redistributive outcome; as we find that $Eg = (12, 4, 5)$, $EgGDP = (21, 25, 26)$, and $p'' = (1.75, 6.25, 5.2)$. However, this greater degree of solidarity is at the expense of the stability outcome, because $Eg = (12, 4, 5) \notin C(N, v_t^0)$.

In the next Figure 2 we draw the core of the tax game (N, v_t^{α}) for the values $\alpha = 0, 0.5, 1$.

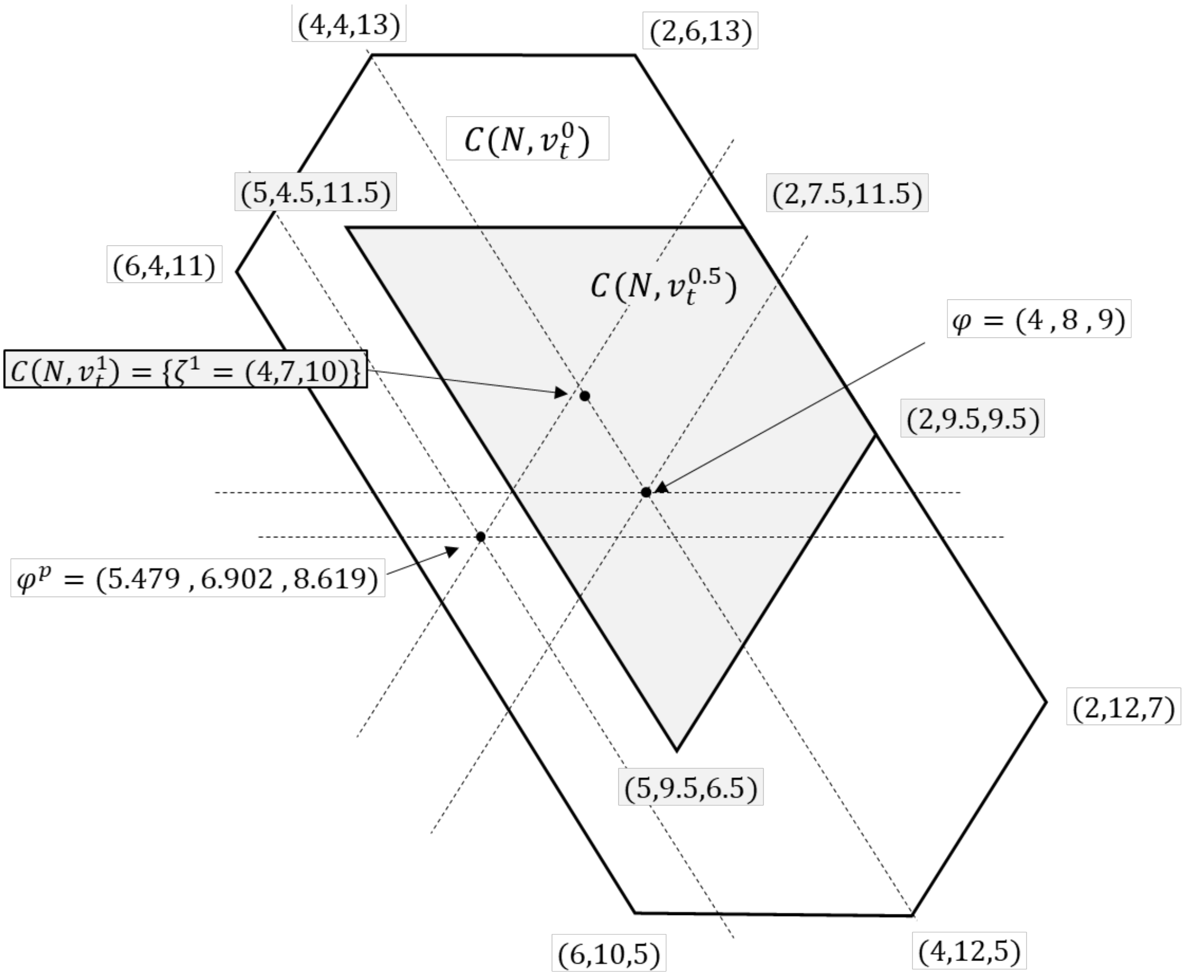

Figure 2

We see in that φ is in the center of $C(N, v_t^0)$ and φ^p is closer to its boundary.

An easy computation yields the following values for the secessionist rules $\zeta^1(N; D, I) = (4, 7, 10)$, and $\zeta^{1/2}(N; D, I) = (3, 5.5, 7.5)$. Clearly, it holds that $\varphi(N; D, I) = (4, 8, 9) \in C\left(N, v_t^{1/2}\right)$, and that $\zeta^{1/2}$ is a lower bound of φ. Moreover, given the differences in the per capita rent between region 1 and regions 2 and 3, we also have that $\varphi^p \notin C\left(N, v_t^{1/2}\right)$.

4 THE SPANISH CASE

In Calvo (2018) the Spanish case was considered by using fiscal data from 2011 to 2014, obtained from the web page "Sistema de Cuentas Públicas Territorializadas" of the "Ministerio Español de Hacienda y Administraciones Públicas"[7]. The data for the inter-regional trade are provided by the c-interg project institution[8], and the data for foreign imports and exports provided by the Spain Ministry of Industry, Commerce and Competitiveness, are available on the Datacomex website[9].

[7] http:// /www.minhafp.gob.es/es-ES/CDI/Paginas/OtraInformacionEconomica/Sistema-cuentas-territorializadas.aspx
[8] http://212.227.102.53/explotacion_multidimensional_comercio_interregional/estadisticas.aspx
[9] http://datacomex.comercio.es/principal_comex_es.aspx

All these data are put together in the Excel file TaxFederalism.xlsx[10]. The reader is referred to Sections 4 and 5 in Calvo (2018), to find a detailed explanation of how these data have been treated.

Spain has 17 Autonomous Communities (AC for short). In what follows we analyze only the data for 2014[11]. The present tax budget spent in each AC is called the *Adjusted Expenditure* AE tax rule. The data of total expenditures are greater than total tax revenues, leading to a fiscal debt for these years. For that reason, we adjust the expenditures in such a way that the total net balance will be zero. In this way, real flows are replaced by "neutralized" flows. For such an operation, we assume that this deficit is redistributed among the ACs in proportion to their population. Next, Table 1 shows the computed data of the tax rules mentioned in the previous sections.

		R	AE	*Eg*	φ	φ^p	ζ^1
Autonomous Communities AC's	AC's	*Total Tax Revenues*	*Adjusted Expenditures*	*Population Egalitarian*	*Balanced Tax Rule*	*Redistributive Balanced Tax Rule*	*Optimistic Seccesionist Rule*
Andalucía	An	53.698.732	61.361.554	67.729.065	50.556.500	52.936.025	51.457.227
Aragón	Ara	11.343.657	12.176.964	10.722.016	13.073.543	13.255.566	12.109.853
Principado De Asturias	Ast	8.650.587	10.746.628	8.508.137	8.850.982	9.151.826	8.733.692
Illes Balears	Ba	9.600.424	8.067.125	9.044.178	10.405.553	10.300.283	10.023.964
Canarias	Cana	12.242.109	16.514.603	17.099.433	13.668.262	14.068.318	12.767.252
Cantabria	Cnt	4.824.547	5.338.254	4.731.993	5.088.824	5.263.242	4.918.681
Castilla y León	C-L	19.142.497	23.410.981	20.061.065	21.068.208	21.711.533	20.046.499
Castilla-La Mancha	C-M	13.785.329	15.411.625	16.689.038	14.733.202	16.095.997	13.932.097
Cataluña	Cat	70.376.301	60.542.524	59.727.914	69.198.108	66.944.452	70.296.839
Comunidad Valenciana	Va	36.155.746	34.436.335	39.933.865	37.802.987	38.526.603	36.524.364
Extremadura	Ex	6.667.215	9.491.444	8.828.966	6.930.511	7.598.951	6.694.142
Galicia	Ga	20.250.535	23.932.122	22.111.274	19.613.170	20.161.802	19.642.107
Comunidad de Madrid	Ma	68.126.895	48.949.930	51.471.680	61.853.122	57.250.828	66.861.289
Región de Murcia	Mu	9.775.540	9.880.471	11.808.044	10.513.588	10.743.165	10.106.533
Comunidad Foral De Navarra	Na	5.731.657	5.847.499	5.133.673	6.143.247	5.996.268	5.811.665
País Vasco	PV	21.052.492	24.449.661	17.478.088	21.698.061	21.092.612	21.301.556
La Rioja	Ri	2.621.104	2.664.702	2.535.172	2.639.780	2.610.658	2.616.199
Ceuta y Melilla	CyMel	929.936	1.752.879	1.361.698	1.137.653	1.267.171	1.131.340
(In thousands euros)	*Total*	*374.975.300*	*374.975.300*	*374.975.300*	*374.975.300*	*374.975.300*	*374.975.300*

Table 1

Firstly, we show that the AE rule is *unstable.* In particular, the payoffs obtained by the AC of Balearics by this rule is not even *individually rational*[12]. In a tax game, we have that $v_t^0(i) = t_{ii}$, and for Balearics it holds that

$$v_t^0(Ba) = 8.449.637 > AE(Ba) = 8.067.125$$

This is a rather extreme situation, because it means that breaking all commercial relationships with the remaining ACs and losing all associated taxes, this community can still manage a greater budgetary amount than it can obtain under the present fiscal system AE. Nevertheless, Balearics is an exception from the point of view of stability. Actually, we are not able to find any other autonomous community, nor a coalition of ACs with positive excess[13]. In particular, Catalonia, which has high secessionist aspirations, has a negative excess:

[10] https://www.uv.es/~ecalvo/TaxFederalism.xlsx
[11] The results obtained for 2011, 2012, and 2013, are very similar, and they can be seen in the TaxFederalism.xlsx file.
[12] Given a TU-game (N, v), a rule ψ is individually rational if $\psi_i(N, v) \geq v(i)$ for all $i \in N$.
[13] Note that the total number of possible coalitions is $2^{18} - 1$, and we have not checked all of them.

$$e(AE, \{Cat\}, v_t^0) = -8.748.568$$

And if we consider the set of 'Catalan Countries', formed by Catalonia (Cat), Valencia (Va), and Balearics (Ba), or the set of 'foral Communities', formed by the Basque Country (PV) and Navarre (Na), their excesses are also negative:

$$e(AE, \{Cat, Va, Ba\}, v_t^0) = -8.699.256$$
$$e(AE, \{PV, Na\}, v_t^0) = -10.210.942$$

In Spain there is a relative wide disparity between the per capita rent of the Autonomous Communities. It goes from €12.066 to €33.247 in 2014. The AC of Madrid (Ma) is 2,76 times richer than the AC of Ceuta and Melilla (CyMel). This big difference in wealth per capita allows us to guess that strong redistributive rules will induce unstable payoffs. As we might expect, we find that the population egalitarian rule Eg is unstable. Indeed, consider the following coalition of ACs that are placed in the bottom part of the per capita GDP range, $S^* = \{CyMel, Ex, An, CM, Cana, Ast, Mu, Ga\}$. The excess of the remaining regions $N \backslash S^*$ is negative:

$$e(Eg, N \backslash S^*, v_t^0) = -4.020.897.$$

Hence, if we delete the set S^* of ACs from the tax system, the remaining ACs, $N \backslash S^*$, will end up in a better position. This means that the transfer of fiscal rents which the Eg rule implies from $N \backslash S^*$ to S^* is enough to make the fiscal system unstable[14].

In order to analyze the redistributive effects of the tax rules, we compute these values in per capita terms in in the following Table 2.

AC's	R_{pc}	AE_{pc}	Eg_{pc}	φ_{pc}	φ_{pc}^{p}	ζ_{pc}^{1}
An	6,400	7,313	8,072	6,025	6,309	6,133
Ara	8,540	9,167	8,072	9,842	9,979	9,117
Ast	8,207	10,195	8,072	8,397	8,682	8,286
Ba	8,568	7,200	8,072	9,287	9,193	8,946
Cana	5,779	7,796	8,072	6,452	6,641	6,027
Cnt	8,230	9,106	8,072	8,680	8,978	8,390
C-L	7,702	9,420	8,072	8,477	8,736	8,066
C-M	6,667	7,454	8,072	7,126	7,785	6,738
Cat	9,511	8,182	8,072	9,352	9,047	9,500
Va	7,308	6,961	8,072	7,641	7,787	7,383
Ex	6,095	8,677	8,072	6,336	6,947	6,120
Ga	7,393	8,736	8,072	7,160	7,360	7,170
Ma	10,684	7,676	8,072	9,700	8,978	10,485
Mu	6,682	6,754	8,072	7,187	7,344	6,909
Na	9,012	9,194	8,072	9,659	9,428	9,138
PV	9,723	11,291	8,072	10,021	9,741	9,838
Ri	8,345	8,484	8,072	8,405	8,312	8,330
CyMel	5,512	10,391	8,072	6,744	7,511	6,706

Table 2

In Table 3, we look at the gross domestic product per capita as the wealth reference. Firstly, we compute what the GDP should be without the fiscal transfers made this year. This is what we call Initial GDP[15]:

[14] The same happens for all $S' \subset S^*$: $v_t^0(N \backslash S') > Eg(N \backslash S')$.

[15] Equivalently, this is the neutralized GDP in which we delete the fiscal debt for the year.

$$IGDP_i = GDP_i - FB_i, \qquad \forall i \in N$$

Final GDP associated to any rule ψ is denoted by ψGDP and is equal to Initial GDP plus the fiscal balance associated with it, that is the payoffs given by the rule ψ to the region minus the total tax revenues of the region, hence:

$$\psi GDP_i = IGDP_i + (\psi_i - R_i), \qquad \forall i \in N$$

In this way, we can compare Initial GDP per capita with Final GDP per capita associated to each ψ.

AC's	*Population*	*IGDP*	*IGDPpc*	*AEGDPpc*	*EgGDPpc*	φ *GDPpc*	φ^p *GDPpc*	ζ^1 *GDPpc*
An	8.390.851	127.635.873	15,211	16,125	16,883	14,837	15,120	14,944
Ara	1.328.334	31.314.451	23,574	24,202	23,106	24,877	25,014	24,151
Ast	1.054.060	18.540.011	17,589	19,578	17,454	17,779	18,065	17,668
Ba	1.120.470	27.522.709	24,564	23,195	24,067	25,282	25,188	24,942
Cana	2.118.423	35.633.414	16,821	18,838	19,114	17,494	17,683	17,069
Cnt	586.240	11.268.218	19,221	20,097	19,063	19,672	19,970	19,382
C-L	2.485.335	47.823.285	19,242	20,960	19,612	20,017	20,276	19,606
C-M	2.067.580	34.639.440	16,754	17,540	18,158	17,212	17,871	16,825
Cat	7.399.601	203.971.107	27,565	26,236	26,126	27,406	27,101	27,554
Va	4.947.346	97.288.007	19,665	19,317	20,428	19,998	20,144	19,739
Ex	1.093.807	13.567.502	12,404	14,986	14,380	12,645	13,256	12,429
Ga	2.739.332	48.885.838	17,846	19,190	18,525	17,613	17,814	17,624
Ma	6.376.749	212.008.327	33,247	30,240	30,635	32,263	31,542	33,049
Mu	1.462.881	25.900.823	17,705	17,777	19,095	18,210	18,367	17,932
Na	636.003	17.286.069	27,179	27,361	26,239	27,826	27,595	27,305
PV	2.165.334	59.245.263	27,361	28,930	25,710	27,659	27,379	27,476
Ri	314.079	7.567.591	24,095	24,233	23,821	24,154	24,061	24,079
CyMel	168.699	2.035.598	12,066	16,945	14,626	13,298	14,066	13,260
Total	*46.455.123*	*1.022.133.527*	*22,003*					

Table 3

We can observe that the present Spanish rule AE has some clear redistributive effects. In general, wealth per capita increases in regions where $IGDPpc$ is lower than average (€22.003), and decreases in regions above average. However, we find some disturbing exceptions. On one hand, we have the case of the Comunidad Valenciana (Va), whose $IGDPpc$ is lower than average; nevertheless, after redistribution, its final per capita rent decreases. On the other hand, regions that are richer than the average end up even richer than initially: La Rioja (Ri), Aragón (Ara), Comunidad Foral de Navarra (Na), and, above all, País Vasco (PV).

The overall redistributive impact is better seen in Table 3, where the maximum and minimum values of the per capita payoffs, and their differences are given for each tax rule, and for their associated final gross domestic product.

Tax Rules	R	ζ^1	φ	φ^p	AE	Eg
Max ψ_{pc}	10,684	10,485	10,021	9,979	11,291	8,072
Min ψ_{pc}	5,512	6,027	6,025	6,309	6,754	8,072
Difference	5,171	4,458	3,995	3,670	4,537	0,000
Max $\psi GDPpc$	33,247	33,049	32,263	31,542	30,240	30,635
Min $\psi GDPpc$	12,066	12,429	12,645	13,256	14,986	14,380
Difference	21,181	20,620	19,619	18,286	15,254	16,255

Table 4

The Fiscal Balances associated to each rule are in Table 4. The first column $R_i - T^i(N)$ is equal to the difference in the total contribution of the remaining regions to the taxes collected by region i minus the total contribution of region i to the taxes collected by the remaining regions. If the sign is positive, region i is in debt with respect to all other regions and it should compensate the system with some monetary transfer. Hence, the tax rule ψ satisfies reciprocity if the fiscal balances $R_i - \psi_i$ have the same sign as $R_i - T^i(N)$.

	Tax contributions	*Fiscal Balances*				
AC's	$R_i - T^i(N)$	$R_i - AE_i$	$R_i - Eg_i$	$R_i - \varphi_i$	$R_i - \varphi_i^p$	$R_i - \zeta_i^1$
An	6.284.464	-7.662.822	-14.030.333	3.142.232	762.706	2.241.505
Ara	-3.459.771	-833.307	621.641	-1.729.886	-1.911.909	-766.196
Ast	-400.789	-2.096.040	142.451	-200.394	-501.239	-83.105
Ba	-1.610.258	1.533.299	556.246	-805.129	-699.859	-423.540
Cana	-2.852.307	-4.272.494	-4.857.324	-1.426.153	-1.826.209	-525.143
Cnt	-528.555	-513.707	92.553	-264.277	-438.695	-94.134
C-L	-3.851.423	-4.268.485	-918.568	-1.925.711	-2.569.037	-904.002
C-M	-1.895.747	-1.626.296	-2.903.709	-947.873	-2.310.668	-146.768
Cat	2.356.385	9.833.776	10.648.387	1.178.193	3.431.849	79.462
Va	-3.294.483	1.719.411	-3.778.119	-1.647.241	-2.370.857	-368.618
Ex	-526.592	-2.824.230	-2.161.752	-263.296	-931.737	-26.928
Ga	1.274.731	-3.681.587	-1.860.739	637.365	88.732	608.428
Ma	12.547.547	19.176.965	16.655.216	6.273.774	10.876.067	1.265.606
Mu	-1.476.096	-104.932	-2.032.504	-738.048	-967.625	-330.993
Na	-823.180	-115.842	597.983	-411.590	-264.611	-80.008
PV	-1.291.139	-3.397.170	3.574.403	-645.570	-40.120	-249.065
Ri	-37.353	-43.599	85.932	-18.676	10.446	4.905
CyMel	-415.436	-822.943	-431.763	-207.718	-337.235	-201.405
	Total transfers	*32.263.452*	*32.974.812*	*11.231.564*	*15.169.800*	*4.199.905*

Table 5

Recall that φ is the only rule built in such a way that reciprocity is satisfied. However, note that φ^p and ζ^1 are also very close to satisfying reciprocity; because the equality of the signs are satisfied for all ACs except La Rioja (Ri). This AC belongs to the group of regions which are in creditor welfare with respect to the rest and this should imply it is a receptor of a tax transfer (negative sign). Although this is a disturbing effect, at least La Rioja belongs to the set of regions with a per capita wealth level greater than the average. The bottom line in Table 5 shows the total transfers made between regions for each tax rule. The population egalitarian Eg is the most redistributive, with total transfers between regions equal to 32.263.452; where the secessionist ζ^1 is the least one, with only 4.199.905 in per capita terms.

We can also visualize the redistributive behavior of the rules by means of a graphic which relates the per capita FB of the rule and the initial gross domestic product. The negative slope of the regression line means an average redistributive effect: lower slope implies a greater redistribution. We can also see the redistributive effect of a tax rule in the slope of the regression line of the graphic which relates the initial GDPpc and the final GDPpc after the application of the tax rule. A positive slope of less than one means a redistributive effect: a lower slope implies a greater degree of redistribution.

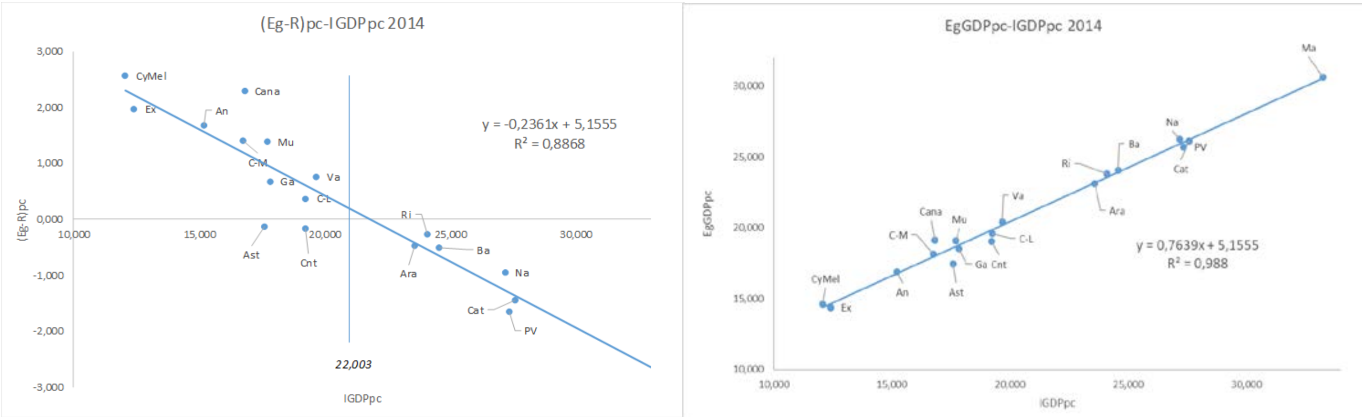

Figure 3. Egalitarian

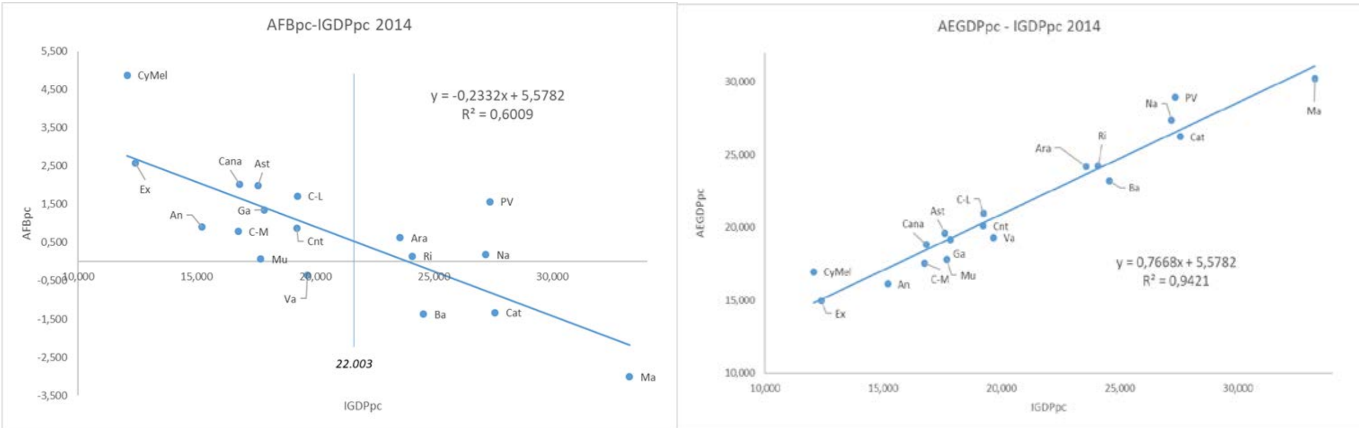

Figure 4. Spanish tax rule

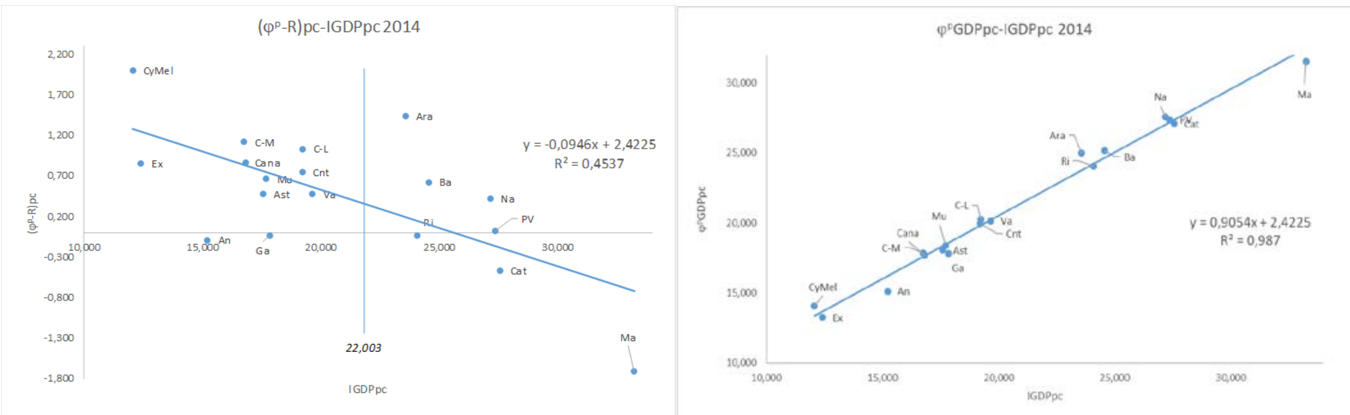

Figure 5. Redistributive balanced

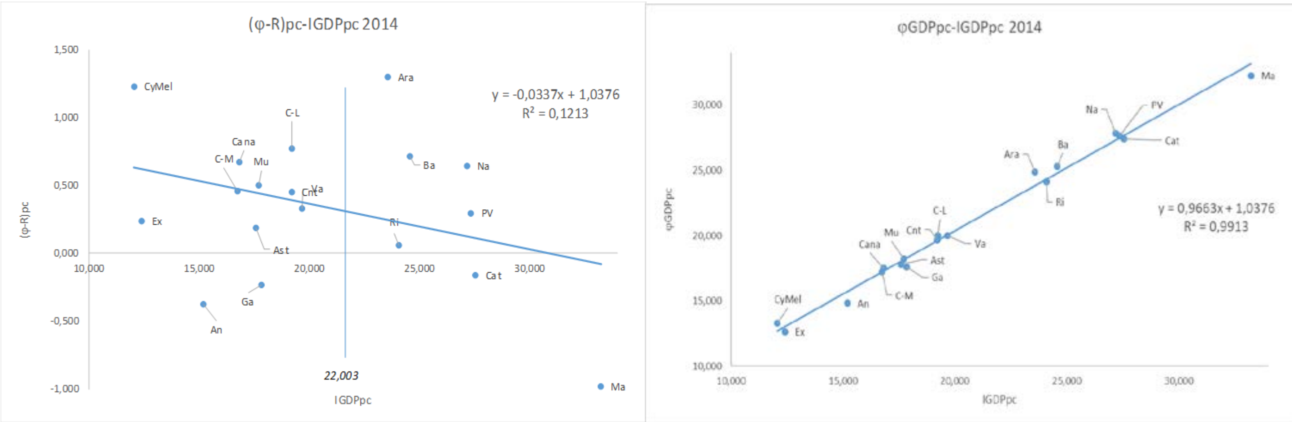

Figure 7. Balanced

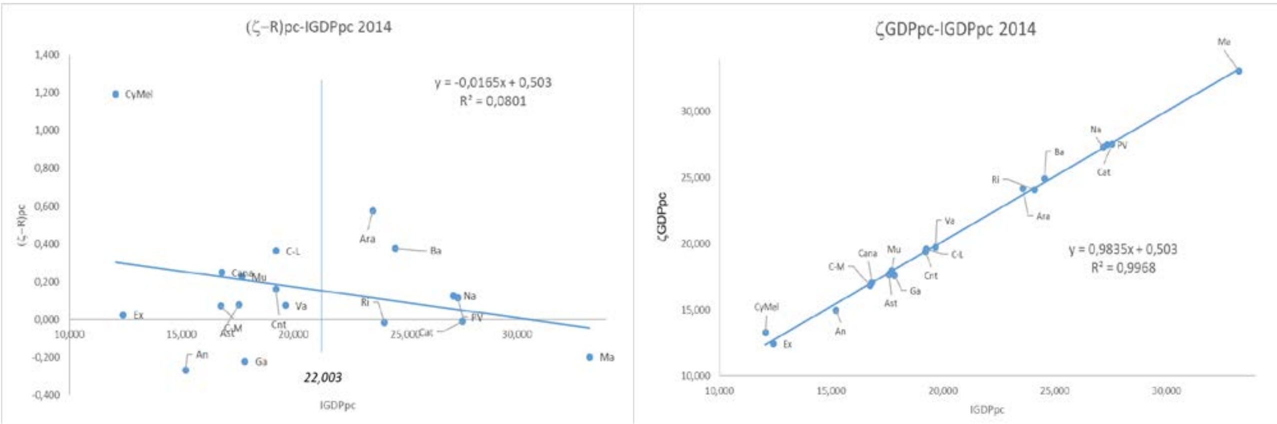

Figure 6. Optimistic secession

There is an additional aspect related with redistribution, and it is the *ordinality* principle of a tax rule:

- The redistributive effect of a rule "should narrow financing disparities across regions without altering their needs-adjusted relative ranking."[16]

In other words, it cannot be that a region i with initial per capita rent greater than region j; i transfers rent to j, and after that, i ends with a lower per capita rent than j. Therefore, a tax rule satisfies ordinality if the relative wealth (in per capita terms) of the regions is preserved after its application. An obvious ordinal tax rule is the no transfer rule ξ, as by definition $\xi_i = T_i(N)$, and then its corresponding fiscal balance is zero. Hence, the GDP remains unaltered.

The Spanish AE rule breaks clearly such a principle: there are 13 pairs of ACs which interchange their wealth position ranking:

$$\begin{Bmatrix} (CyMel, Ex), (CyMel, An), (Cana, Mu), (Ast, Mu), (Ast, Ga), (Ast, Va), (Ga, Va), \\ (Cnt, Va), (CL, Va), (Ara, Ba), (Ri, Ba), (Na, Cat), (PV, Cat) \end{Bmatrix}$$

For the population egalitarian rule Eg the are 10 pairs of ACs which change their wealth ranking:

$$\begin{Bmatrix} (CyMel, Ex), (CM, Ast), (Mu, Ga), (Cana, Ast), (Cana, Mu), \\ (Cana, Ga), (Mu, Ga), (Mu, Cnt)(Na, PV), (Na, Cat) \end{Bmatrix}$$

[16] De la Fuente, Thöne and Kastrop (2016).

φ^p exhibits the same behavior as φ, because both have the same eight pairs of changes in wealth ranking:

$$\{(CyMel, Ex), (Ast, Ga), (Mu, Ga), (CL, Va), (Ara, Ri), (Na, PV), (Na, Cat), (PV, Cat)\}.$$

Finally, ζ^1 has only four pairs of changes.

$$\{(CyMel, Ex), (Ast, Ga), (Mu, Ga), (Ara, Ri)\}$$

We can observe an indirect relationship between solidarity (wealth redistribution) and ordinality principles. Given the unequal wealth distribution in Spain, the most solidaritarian rules considered here, Eg and AE, turn out to be unstable, even in the most pessimistic scenario $\alpha = 0$. Moreover, they exhibit a bad behavior from the ordinality point of view. In the opposite direction, φ^p, φ, and ζ^1, work better from the stability and ordinality side, although at the cost of being more unsupportive. The redistributive tax rule φ^p is a tradeoff between these two opposite sides: stability and ordinality on one hand, solidarity/redistribution on the other one.

5 COOPERATIVE VALUES IN TAX GAMES

The property of the stability of an allocation rule ψ in a tax problem $(N; D, I)$ was given in terms of its associated tax game (N, v_t^α), in the sense that ψ is stable if, and only if, its payoffs belong to the core of its corresponding tax game: $\psi(N; D, I) \in C(N, v_t^\alpha)$. The core is a set value function. It defines the set of stable allocations in a game. So we have identified the set of stable tax allocations of a tax problem $(N; D, I)$ by the set of stable allocations of its associated tax game (N, v_t^α).

In the Literature on cooperative games many different solutions had been defined that are single-valued. Therefore, an obvious question that immediately arises is the translation of values defined in the setting of TU-games into the setting of tax problems, by using its associated tax game. So, given a value ψ defined in $\mathcal{G}$, the value can be extended to $\mathcal{T}$ simply by defining for each $(N; D, I) \in \mathcal{T}^N$ the tax rule ψ by $\psi(N; D, I) \coloneqq \psi(N, v_t)$.

For example, the population egalitarian rule Eg has its corresponding counterpart with the weighted split solution ε^w defined by

$$\varepsilon^w(N, v) = \frac{w_i}{w(N)} v(N),$$

where $w \in \mathbb{R}^N_{++}$. If we consider weights as population of regions, it is immediate that, for all α, it holds that

$$\varepsilon^w(N, v_t^\alpha) = \frac{w_i}{w(N)} v_t^\alpha(N) = \frac{w_i}{w(N)} T(N) = Eg(N; D, I).$$

The reader familiar with the Shapley value (Shapley, 1953a,b), will already have guessed that the balanced tax rule φ is just the translation into the tax problems setting of the Shapley value, Sh.

We briefly recall the definition of Sh. Let a game (N, v), and let $\pi: N \to N$ be a permutation of the player set N. Denote by $\Pi(N)$ the set of all permutations defined in N. We interpret π as an order defined between players in N, i.e. each player i enter in position $\pi(i)$ in the order π. Denote by $P_\pi(i)$ the set of *predecessors of i in order π*, that is

$$P_\pi(i) \coloneqq \{j \in N\colon\ \pi(j) < \pi(i)\}.$$

The *marginal contribution* that each player $i \in N$ receives in every order $\pi \in \Pi(N)$ is given by

$$m_i^\pi(N, v) = m_i(P_\pi(i), v) = v(P_\pi(i) \cup i) - v\big(P_\pi(i)\big).$$

and define the *marginal vector* $m^{\pi}(N,v) \in \mathbb{R}^N$ by $m_i^{\pi}(N,v) := m_i(P_{\pi}(i), v)$, for all $i \in N$. It is immediate that

$$\sum_{i \in N} m_i^{\pi}(N,v) = v(N).$$

The *Shapley value* (Shapley, 1953a,b) of the game is defined as:

$$Sh_i(N,v) = \frac{1}{n!} \sum_{\pi \in \Pi(N)} m_i^{\pi}(N,v), \qquad \forall i \in N.$$

The value $Sh_i(N,v)$ is the *expected marginal contribution* of player i with respect to the uniform distribution over all orders on N. As we will see in the Appendix with more detail, it holds that $\varphi(N; D, I) = Sh(N, v_t^0)$.
Another relevant value in the purpose to find stable tax allocations is the nucleolus (Schmeidler, 1969). The nucleolus was introduced in Schmeidler (1969). The idea behind this value is to select an allocation which minimizes the dissatisfaction of the most dissatisfied coalition, where the dissatisfaction of an allocation x by coalition $S \subseteq N$ is measured by its excess $e(S,x,v)$. For that purpose, it makes the largest dissatisfaction as small as possible. If there are several allocations to do this, then make the second largest dissatisfaction as small as possible, etc., until a unique allocation is reached. In this sense, the nucleolus is similar in spirit to the *maxmin principle* of distributive justice proposed in Rawls (1971).
Given an allocation x, denote by $\theta(x)$ the *excess vector* with respect to x. $\theta(x)$ is the vector in $\mathbb{R}^{2^n-1}$ which contains the excesses of all coalitions in (weakly) decreasing order. Let two excess vectors $\theta(x), \theta(y) \in \mathbb{R}^{2^n-1}$, we say that $\theta(x)$ is *lexicographically smaller* than $\theta(y)$, $\theta(x) \leq_L \theta(y)$, if either $\theta(x) = \theta(y)$, or there is $i \in \{1,2,\ldots,2^n - 1\}$ such that $\theta_j(x) = \theta_j(y)$, for each $j < i$, and $\theta_i(x) < \theta_i(y)$. The *imputation set* of a game (N,v) is defined by the set of all efficient and individually rational allocations. That is,

$$I(N,v) = \{x \in \mathbb{R}^N : x(N) = v(N) \quad \wedge \quad x_i \geq v(i), \quad \forall\, i \in N\}.$$

The nucleolus of a game, $\eta(N,v)$, is the set

$$\eta(N,v) = \{x \in I(N,v): \ \theta(x) \leq_L \theta(y), \forall y \in I(N,v)\}.$$

When $I(N,v) \neq \emptyset$, the nucleolus selects a *unique* point in the imputation set. Moreover, the nucleolus is always in the core when the core is non-empty. There are several procedures to compute the nucleolus, but it can be quite hard. See Maschler (1992) for a review of those procedures. For practical computations, the Matlab toolbox package *MatTuGames* provided by Meinhardt (2012) can be used.
It should be noticed that, in general, the Shapley value and the nucleolus will select different points, even for convex games. So this fact leads us to expect that the nucleolus and the balanced allocation of a tax game should be different. But surprisingly, it turns out that both values coincide in pessimistic tax games, that is, $Sh(N, v_t^{\alpha}) = \eta(N, v_t^{\alpha})$.

We know at least an additional coincidence with a different solution concept. This is the τ-value (Tijs, 1981). It selects a compromise between some maximal and minimal payoff aspirations that players have in a game. These upper and lower bounds are built as follows.
Let a game (N,v) and define the vector $M(v) \in \mathbb{R}^N$ by

$$M_i(v) = v(N) - v(N \setminus i), \qquad \forall i \in N.$$

Each component $M_i(v)$ is called the *utopia payoff* for player i, and it is the maximal payoff which player i can expect to get, because if i claims more than $M_i(v)$, the rest of players will

obtain less than $v(N\backslash i)$. In such a case it is advantageous for $N\backslash i$ to exclude him from the agreement. In that sense, $M(v)$ is an upper payoff bound of v.
Let a coalition $S \subseteq N$ and player $i \in S$, the *remainder of* i in S is defined by

$$R^v(S,i) = v(S) - \sum_{j \in S\backslash i} M_j(v).$$

The amount $R^v(S,i)$ is what remains for i if S forms and the rest of players in S obtain their utopia payoff. The lower bound of v is given by

$$m_i(v) = \max_{S: i \in S} R^v(S,i).$$

The amount $m_i(v)$ is called the *minimal right* of player i, as i can guarantee himself this payoff offering the rest of members of such maximal coalition S, their utopia payoffs and taking the rest for himself. A game (N,v) is *quasi-balanced* if $m(v) \leq M(v)$, and

$$\sum_{i \in N} m_i(v) \leq v(N) \leq \sum_{i \in N} M_i(v).$$

For the class of quasi-balanced games the τ-value is defined as a *compromise* between the upper $M(v)$, and lower $m(v)$, bounds of the game, that is:

$$\tau(N,v) := m(v) + \alpha\big(M(v) - m(v)\big),$$

where α is such that $\sum_{i \in N} \tau_i(N,v) = v(N)$.
We remark that the τ-value does not necessarily belong to the core, even for convex games. However, for the pessimistic tax game (N, v_t^0) it turns out that the τ-value coincides with the balanced allocation $\varphi(N;D,I)$. This coincidence implies that it also belongs to the core.

We summarize these coincidences in the next theorem.

Theorem 25 *For every tax problem* $(N;D,I) \in \mathcal{T}^N$ *it holds that* $\varphi(N;D,I) = Sh(N,v_t^0) = \eta(N,v_t^0) = \tau(N,v_t^0)$.

This is a quite remarkable result, because the approach that supports each of these values is based on very different ideas, so nothing makes us think that such a coincidence should happen in the class of tax games.

We find a different behavior for the redistributive balanced tax rule φ^p. It is immediate that with the vector of weights $\omega = (p_i^{-1})_{i \in N}$ the redistributive balanced tax rule is a weighted Shapley value of its tax game: $\varphi^p(N;D,I) = Sh^\omega(N,v_t^0)$. However there is not an equivalent result for a weighted version of the nucleolus and the τ-value.

It is noteworthy that rules defined in the setting of tax problems $\mathcal{T}$ have not a counterpart in the setting of TU-games $\mathcal{G}$. For example, the no transfers rule ξ, defined by $\xi_i(N;D,I) = T_i(N)$, for all $i \in N$, cannot be defined by only taking into consideration the information provided by its tax game (N,v_t^0). The procedure followed by ξ is to assign the sum of coefficients of file i in matrix T to player i. This procedure can be generalized assigning an arbitrary set of coefficients of the matrix D and I to each player, such that each coefficient belongs to only one player and all coefficients are distributed among players. Denote by $\mathcal{P}$ such partition and let $\xi_i^{\mathcal{P}}(N;D,I)$ the sum of the coefficients assigned to player i by partition $\mathcal{P}$. In the same way, we cannot find its corresponding value applied to (N,v_t^0).

Finally, when $\alpha = 1$ the set of stable allocations is a single point, i.e. $C(N,v_t^1) = \{\zeta^1(N;D,I)\}$.

As (N, v_t^1) is an additive game, it is clear that the Shapley value, the nucleolus, and the τ-value, all of them coincide in the same optimistic secession point. However, this equivalence for tax games (N, v_t^α) is no longer true for the case of $0 < \alpha < 1$. Take the numerical Example in subsection 3.4 , and consider the case where $\alpha = {}^1/_2$. The associated tax game $\left(N, v_t^{1/2}\right)$ is given by

$$v_t^{1/2}(1) = 2, v_t^{1/2}(2) = 4, v_t^{1/2}(3) = 5, v_t^{1/2}(\{1,2\}) = 9.5,$$

$$v_t^{1/2}(\{1,3\}) = 11.5, v_t^{1/2}(\{2,3\}) = 16, v_t^{1/2}(\{1,2,3\}) = 21.$$

We can check that

$$Sh\left(N, v_t^{1/2}\right) = (3.5, 7.75, 9.75),$$
$$\eta\left(N, v_t^{1/2}\right) = (4.3333, 7.5833, 9.0833),$$
$$\tau\left(N, v_t^{1/2}\right) = (3.8462, 7.5769, 9.5769).$$

6 FINAL REMARKS

The analytic tool of the cooperative tax game v_t^α is presented. Its purpose is to give an accurate measure of the tax budget which a region or group of regions by themselves can manage, without the help of the remaining regions, in case of secession.

The parameter α specifies the optimistic/pessimistic perception of the economic consequences of secession. Specifically, the intensity of the border effect in the commercial trade relationships. The range of values goes from the most pessimistic scenario $\alpha = 0$, in which all trade relations are broken, to the most optimistic $\alpha = 1$, where everything remains equal, except for the emergence of a new country with its new borders.

The core of the tax game (N, v_t^α) allow us to evaluate the stability of a present tax rule, or any alternative proposal, as for example, the population egalitarian rule Eg and the optimistic secession rule ζ^1. Both tax rules were already considered in the former paper Calvo (2018). ζ^1 is stable for all α, and Eg typically is unstable even for $\alpha = 0$ when there are big differences in the per capita rent among regions.

In this paper we present two new tax rules: the *balanced* tax rule φ, and the *p-balanced* tax rule φ^p.

The idea behind φ is to balance the contribution of the welfare that each region makes to each other in form of tax transfers. Three different solutions: the *Shapley value* (Shapley, 1953a,b), the *nucleolus* (Schmeidler, 1969), and the τ-value (Tijs, 1981), when they are applied to the tax game (N, v_t^0), all of them coincide with φ. As the tax game is convex by construction, it is *stable*. Indeed, as φ is the nucleolus of (N, v_t^0), it means that is the most stable one for $\alpha = 0$. Moreover, it is always stable for any $\alpha \leq 1/2$.

When there are wide differences in the per capita wealth between regions, a desirable characteristic of a regional financial system should be the solidarity among regions, in the form of some degree of rent transfers from richer to poorer regions. In the case of wide disparities in per capita rent among regions, φ usually exhibits a poor redistributive behavior. For this reason, we present the *p-balanced tax* rule φ^p, where the weights p are the per capita wealth of the regions. Now the welfare contribution that each region makes to each other in form of tax transfers is balanced inversely proportional to their per capita rent. φ^p coincides with the weighted Shapley value (Shapley, 1953a) on the tax game (N, v_t^0), and it is stable for $\alpha = 0$. By its construction, φ^p offers a greater degree of solidarity among regions than φ.

The Spanish case has been used to illustrate how to apply all these concepts. Thus, Calvo (2018) shows how to build the tax matrices $[I]$, $[D]$, and $[T]$, and its associated tax game v_t^α. Here, both tax rules φ and φ^p are also computed. In addition we compare their behavior with of the present Spanish rule AE, the population egalitarian Eg, and the optimistic secessionist ζ^1.

We can observe an indirect relationship between solidarity (rent redistribution) and ordinality principles (preserving rank per capita rent). Given the unequal wealth distribution in Spain, the most solidaritarian rules considered here are Eg and AE. However, it is at the cost of being unstable, even for the most pessimistic scenario $\alpha = 0$. Moreover, they exhibit a bad behavior from the ordinality point of view. In the opposite direction, φ^p, φ, and ζ^1, works better from the stability and ordinality side, although at the cost of being more unsupportive. We believe that the redistributive tax rule φ^p is a reasonable tradeoff between these two opposite sides: stability and ordinality on the one hand, solidarity on the other one.

There are some issues that would be of interest to further research.

The first one is obvious, we can check the stability of the financial regional system in any country with available data for taxes and inter-regional commercial trade between their regions.

Secondly, there is a computational problem checking whether a tax allocation belongs to the core of the tax game (N, v_t^0). The size of the subcoalitions of N is 2^n, which increases exponentially with n. It would be of interest to find efficient algorithms to check whether an allocation belongs to the core in polynomial time, helping with the convexity and simplicity of v_t^0.

Finally, it would be of interest to find the most population egalitarian allocation inside the core of (N, v_t^0). That is, a translation into the setting of tax problems of the Dutta-Ray egalitarian solution (see Dutta and Ray, 1989).

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9 APPENDIX

9.1 Proof of Theorem 25

We decompose the proof of Theorem 25 into three propositions. One for each value.

9.1.1 The Shapley value

Apart from the original Shapley's characterization of this value (Shapley, 1953a,b), several other characterizations have been outlined in the Literature. For example, Myerson (1980), Young (1985), Hart and Mas-Colell (1989), Feltkamp (1995), and van den Brink (2001), among others. We notice that the property of balanced allocations, which we have used in the setting of tax problems is just a forward translation of the property of balance contributions introduced by Myerson (1980), for TU-games. With efficiency and balanced contributions, Myerson gave one of the simplest axiomatic characterization of this value. We recall here briefly his result.

Definition 26 *We say that a solution* ψ *on* $\mathcal{G}$ *satisfies balanced contributions if* $\psi_i(N,v) - \psi_i(N\backslash j,v) = \psi_j(N,v) - \psi_j(N\backslash i,v)$ *for every pair of players* $\{i,j\} \subseteq N, i \neq j$*, and every game* (N,v).

The Myerson's characterization is:

Theorem 27 *(Myerson, 1980) There exists a unique value on $\mathcal{G}$ which satisfies efficiency and balanced contributions. This is the Shapley value.*

Given our characterization of φ by efficiency and balanced allocations on tax problems, it is immediate the following equivalence:

Proposition 28 *For every tax problem $(N; D, I) \in \mathcal{T}^N$ it holds that $\varphi(N; D, I) = Sh(N, v_t^0)$.*

Proof Let $(N; D, I)$ be a tax problem and (N, v_t^0) its associated tax game. Define $\varphi(N, v_t^0) \coloneqq \varphi(N; D, I)$. As

$$\sum_{i \in N} \varphi_i(N, v_t^0) = \sum_{i \in N} \varphi_i(N; D, I) = T(N) = v_t^0(N),$$

it holds that φ satisfies efficiency.

For every $i \in N$, the sub tax problem $\left(N\backslash i; D^{-i}, I^{-i}\right)$ has the associated subgame $(N\backslash i, v_t^0)$. Therefore $\varphi(N\backslash i, v_t^0) \coloneqq \varphi\left(N\backslash i; D^{-i}, I^{-i}\right)$. Let $\{i, j\} \subseteq N$, then

$$\varphi_i(N, v_t^0) - \varphi_i(N\backslash j, v_t^0) = \varphi_i(N, v_t^0) - \varphi_i\left(N \setminus j, D^{-j}, I^{-j}\right) = \frac{1}{2}\left(t_{ij} + t_{ji}\right)$$
$$= \varphi_j(N, v_t^0) - \varphi_j\left(N \setminus i, D^{-i}, I^{-i}\right) = \varphi_j(N, v_t^0) - \varphi_j(N \setminus i, v_t^0).$$

hence φ satisfies balanced contributions. Then, by Theorem (40), we find that $Sh(N, v_t^0) = \varphi(N, v_t^0) = \varphi(N; D, I)$ ■

In Hart and Mas-Colell (1989, Section 5) the weighted Shapley values, Sh^ω were considered. Given a vector of weights ω, a weighted version of balanced contributions property is as follows:

Definition 29 *Given a vector $\omega \in \mathbb{R}_{++}^N$, a value ψ on $\mathcal{G}$ satisfies weighted balanced contributions if $\frac{1}{\omega_i}\left(\psi_i(N, v) - \psi_i(N\backslash j, v)\right) = \frac{1}{\omega_i}\left(\psi_j(N, v) - \psi_j(N\backslash i, v)\right)$ for every pair of players $\{i, j\} \subseteq N, i \neq j$, and every game (N, v).*

And it was proved:

Theorem 30 *(Hart and Mas-Colell, 1989) Given a vector $\omega \in \mathbb{R}_{++}^N$, there exists a unique value on $\mathcal{G}$ which satisfies efficiency and weighted balanced contributions. This is the weighted Shapley value Sh^ω.*

Now consider the tax rule φ^p. The vector of weights $p \in \mathbb{R}_{++}^N$ is given by the initial per capita richness of the region, i.e. $p_i = \frac{IGDP_i}{w_i}$, for all $i \in N$. Defining $\omega_i = p_i^{-1}$, it is immediate that if a tax rule verifies the p-balanced tax property, for weights $(p_i)_{i \in N}$, in the tax problem $(N; D, I)$ it also verifies weighted balanced contributions, for weights $(\omega_i = p_i^{-1})_{i \in N}$, in the tax game (N, v_t^0). Then it is straightforward to obtain a parallel result that of Proposition 29 and hence we omit the proof.

Theorem 31 *For every tax problem $(N; D, I) \in \mathcal{T}^N$ and weights $p \in \mathbb{R}_{++}^N$ it holds that $\varphi^p(N; D, I) = Sh^\omega(N, v_t^0)$, where $(\omega_i = p_i^{-1})_{i \in N}$.*

9.1.2 The nucleolus

The proof that the balanced allocation is also the nucleolus of its associated tax game is made by showing that the balanced allocation satisfies the Kohlberg criterion (Kohlberg, 1971). We need some preliminary definitions.

Let $\lambda = (\lambda(S))_{S\subseteq N}$, we say that λ is a *balanced collection of weights* if

$$\begin{cases} 1)\ \lambda(S) \geq 0, & \forall S \subseteq N, \\ 2) \sum_{S\subseteq N: i\in S} \lambda(S) = 1, & \forall i \in N. \end{cases}$$

A collection B of coalitions is called a *balanced collection* if there is a balanced collection of weights λ such that $B = \{S \subseteq N\colon\ \lambda(S) > 0\}$. Let $\alpha \in \mathbb{R}$ and $x \in \mathbb{R}^N$, we denote by $D(\alpha, x, v) = \{S \in 2^N \backslash \emptyset\colon e(S, x, v) \geq \alpha\}$. The following characterization was given originally for the pre-nucleolus of a game. However, the pre-nucleolus and the nucleolus coincide when the core is nonempty. As this is the case for tax games, we will use this version of the Kohlberg criterion.

Theorem 32 *(Kohlberg, 1971) Let (N, v) be a game with $C(N, v) \neq \emptyset$, and $x \in C(N, v)$. Then the following two statements are equivalent:*
K.1) $x = \eta(N, v))$,
K.2) $D(\alpha, x, v)$ is a balanced collection for every $\alpha \in \mathbb{R}$ with $D(\alpha, x, v) \neq \emptyset$.

Now we can establish the equivalence between the balanced allocation and the nucleolus:

Proposition 33 *For every tax problem $(N; D, I) \in \mathcal{T}^N$ it holds that $\varphi(N; D, I) = \eta(N, v_t^0)$.*

Proof Let $(N; D, I)$ be a tax problem, and let $x = \varphi(N; D, I)$. Firstly, note that, for all $S \in 2^N \backslash \emptyset$, the excess of a coalition and its complementary is the same:

$$e(S, x, v_t^0) = -\frac{1}{2}\sum_{i\in S}\sum_{j\in N\backslash S}\left(t_{ij} + t_{ji}\right) = e(N \setminus S, x, v_t^0). \tag{9}$$

As we know that $x \in C(N, v_t^0)$, if $\alpha > 0$ then $D(\alpha, x, v_t^0) = \emptyset$. Now let $\alpha \leq 0$. Then it holds that $N \in D(\alpha, x, v_t^0)$ because $e(N, x, v_t^0) = 0 \geq \alpha$. Denote by $d = |D(\alpha, x, v_t^0)|$. Therefore, the number d must be odd, because if there is $S \neq N$ such that $S \in D(\alpha, x, v_t^0)$ then it happens also that $N\backslash S \in D(\alpha, x, v_t^0)$. This implies that each $i \in N$ belongs to $\frac{d-1}{2} + 1$ coalitions which are in $D(\alpha, x, v_t^0)$. Now, let the collection of weights λ given by

$$\lambda(S) = \begin{cases} \dfrac{1}{\frac{d-1}{2} + 1}, & if\ \ S \in D(\alpha, x, v_t^0), \\ 0, & if\ \ S \notin D(\alpha, x, v_t^0). \end{cases}$$

The collection of weights λ is balanced by construction, and then $D(\alpha, x, v_t^0)$ forms a balanced collection of coalitions. Therefore, by Theorem (33) we deduce that $\varphi(N; D, I) = \eta(N, v_t^0)$ ■

There is not an evident way to find a weighted version of the nucleolus, η^ω, such that $\eta^\omega(N, v_t^0) = \varphi^p(N; D, I)$, for some $\omega = f(p)$.

9.1.3 The $\boldsymbol{\tau}$-value

We use the fact that tax games are convex to prove the equivalence of the balanced tax rule and the τ-value

Proposition 34 *For every tax problem $(N; D, I) \in \mathcal{T}^N$ it holds that $\varphi(N; D, I) = \tau(N, v_t^0)$.*

Proof Let $(N; D, I)$ be a tax problem and (N, v_t^0) its associated tax game. It is known that if a game v is convex it holds that $m_i(v) = v(i)$ for all $i \in N$. As the tax game (N, v_t^0) is convex, we have that $m_i(v_t^0) = t_{ii}$. Moreover, we have that

$$M_i(v_t^0) = v_t^0(N) - v_t^0(N \setminus i) = t_{ii} + \sum_{j \in N \setminus i} \left(t_{ij} + t_{ji}\right).$$

and then

$$\tau_i(N, v_t^0) = t_{ii} + \alpha \left(\left(t_{ii} + \sum_{j \in N \setminus i} \left(t_{ij} + t_{ji}\right) \right) - t_{ii} \right) = t_{ii} + \alpha \sum_{j \in N \setminus i} \left(t_{ij} + t_{ji}\right).$$

By efficiency, it holds that $v_t^0(N) = \sum_{i \in N} \tau_i(N, v_t^0)$. This implies that $\alpha = \frac{1}{2}$, and then $\tau(N, v_t^0) = \varphi(N; D, I)$ ■

As in the case of the nucleolus, we have the same difficulties to find a suitable weighted version of the τ-value which coincides with the p-balanced tax rule. Given two vectors $x, y \in \mathbb{R}$, the scalar product is denoted by $x * y = (x_i y_i)_{i \in N}$. For a weight vector $\omega \in \mathbb{R}^N_{++}$, the weighted τ-value can be defined by

$$\tau^\omega(N, v) = m(v) + \alpha\omega * \left(M(v) - m(v)\right),$$

where $\alpha \in \mathbb{R}$ is such that $\sum_{i \in N} \tau_i^\omega(N, v) = v(N)$. Again, there is not an easy way to build a vector ω such that $\tau^\omega(N, v_t^0) = \varphi^p(N; D, I)$, for some $\omega = f(p)$.

Theorem 25 is a corollary of Propositions 29, 34, and 36.

9.2 Review of the Literature of equivalence

For the sake of completeness, we review briefly the Literature of equivalence between the Shapley value and the nucleolus. In Brown and Housman (1988) it was given a sufficient condition for the coincidence of $Sh = \eta = \tau$. Such condition actually implies that the excess of a coalition and its complementary are the same. The family of pessimistic tax games satisfies this sufficient condition (see equation (9)). For this reason, the proof of the equivalence between φ and the nucleolus is basically the same as in Brown and Housman. However, for the Shapley value and the τ-value our proof is different to that of Brown and Housman.

Notice that (N, v_t^0) can be seen as a cooperative game defined in an extended digraph. That is, given a weight function $t: N \times N \to \mathbb{R}_+$, the cooperative game v_t^0 is defined by $v_t^0(\emptyset) = 0$ and

$$v_t^0(S) = \sum_{(i,j) \in S \times S} t(i,j), \qquad \forall S \subseteq N, S \neq \emptyset.$$

In an extended digraph, the pairs (i, j) and (j, i), with $i \neq j$, are considered as different elements of the digraph, and pairs (i, i) also belong to the digraph. The family of weighted graphs considered in Brown and Housman (1988), is the particular case of undirected graphs, where pairs (i, j) and (j, i), with $i \neq j$, are considered as the same element of the graph, and pairs (i, i) are not allowed. Weighted graphs belong to the family of so called 2-players games, characterized by $v(S) = 0$ if $|S| \leq 1$, and $v(S) = \sum_{T \subseteq S, |T|=2} v(T)$ if $|S| \geq 2$.

Kar *et al.* (2009) define a family of games called PS games. A game (N, v) is a PS game if for all $i \in N$, there exists $c_i \in \mathbb{R}$ such that $v(S \cup i) - v(S) + v(N \setminus S) - v\left(N \setminus (S \cup i)\right) = c_i$ for all $S \subseteq N \setminus i$. This property says that the sum of a player's marginal contribution to any coalition S and its complement $N \setminus (S \cup i)$ is a player specific constant. They prove that in the family of PS games, the Shapley value and the nucleolus coincide. Moreover it holds that $Sh_i(N, v) = \eta_i(N, v) = \frac{c_i}{2}$, for all $i \in N$. Any 2-player game is a PS game, but not *vice versa*. A pessimistic tax game is not a 2-player game, because $v_t^0(i) = t_{ii} \neq 0$, however, it is a PS game, as it is easy to check that

$$v_t^0(S \cup i) - v_t^0(S) + v_t^0(N \backslash S) - v_t^0\big(N \backslash (S \cup i)\big) = 2t_{ii} + \sum_{j \in N \backslash i} \big(t_{ij} + t_{ji}\big)$$

for all $S \subseteq N \backslash i$. Again, the proof of the equivalence between Sh and η in PS games given in Kar *et al.* is made in a different way to our proof.