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Abstract

We present a general bargaining protocol between n players in the setting of coalitional games with transferable utility. We consider asymmetric players. They are endowed with different probabilities of being chosen as proposers and with different probabilities of leaving the game if offers are rejected. Two particular specifications of this bargaining protocol yield equilibrium proposals that we refer to as *weighted solidarity values* and *weighted Shapley values*. We compare the behavior of these values when the players' probabilities are changed. We supplement the analysis with axiomatic characterizations of both values.

Keywords: n-person bargaining; transferable utility games; asymmetric players; solidarity value; Shapley value.

JEL Classification: C71

1 Introduction

In a cooperative setting, a value expresses a particular way in which players[1] share the benefits of their cooperation. Following the *Nash program,* one can determine a particular value either through a set of properties that the value satisfies (the axiomatic approach) or through a non cooperative game which reflects a plausible negotiation process (the strategic approach). In the latter case cooperative agreement is obtained as the equilibrium payoffs of the non cooperative game. The two approaches are considered as complementary and hopeful of mutual reinforcement.[2]

Most of the bargaining models proposed in the literature have a nice interpretation as protocols that can implement cooperative solution concepts in a decentralized way. The goal with this approach is the cooperative solution, and the model is a means of finding it. A different option is to present a simple, natural bargaining process and determine the payoffs that the players involved in that process can expect to obtain. The bargaining model considered here follows this second approach. In it, players make offers

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[1] From now on, we interpret players in a game as agents with neutral gender. They can be interpreted as automata, institutions, firms, political parties or so on. Therefore we will avoid choosing their gender every time.

[2] For a good survey on the Nash program readers are referred to Serrano (2005).

and counteroffers up to the time when a proposal is accepted. Each player can make offers and the selection of the proposer is a random process. If the proposal is rejected by even one player, another proposer (or perhaps the same one) is selected randomly until an offer is accepted by the remaining players. All players start the negotiations at the beginning.

Additionally, we assume that time is costly, in the sense that the players' time is not infinite. As long as players reject offers the time spent in bargaining increases, and this process cannot be continued indefinitely. Therefore, we assume that the probability of a player leaving the negotiation increases as time goes by. When a player leaves the game, it receives a payoff of zero and the remaining players restart the bargaining process without it.

This is a noncooperative game which has stationary subgame perfect equilibria. In Theorem 1 in Section 2, we show that equilibrium proposals are easily characterized: The proposing player offers its continuation value to every active player and claims the rest of the pie for itself.

In general, players are not completely identical in their bargaining skills or in the circumstances under which they are involved. In the present study we introduce two type of asymmetry which are related to specific parameters of bargaining. The first parameters, $\alpha = (\alpha_i)_{i \in N}$, are related to the different probability that each player i has of being the proposer. The source of this asymmetry could lie in differences in bargaining skills, or in players being representatives of groups (teams, parties, cities, countries, etc) of different sizes. The second parameters, $\omega = (\omega_i)_{i \in N}$, are related to the different probability that each player i has of leaving the game after the rejection of a proposal. This second type of asymmetry seeks to capture differences in fitness characteristics, such as different expected time of life or vitality.

In Section 3 we analyze two families of asymmetric values which arise from two particular specifications in the bargaining protocol considered in Section 2. They only differ in what happens when a proposal is rejected. In the first case every player who is still in the game has its own independent probability of leaving the game. In the second case, the probabilities of leaving the game depend on who the proposer is. In particular, only the proposer has its own probability of leaving, and the rest of the players who are still in the bargaining remain in the game for sure.

As negotiations take place over time, we consider a time interval of fixed length $\Delta > 0$ between consecutive offers. Theorem 2 considers the first case (all players can leave the game) and Theorem 3 considers the second (only the proposer has the chance to leave). In both cases, for each specification of the players' probabilities of being the proposer and leaving the game the limit of the equilibrium proposals can be calculated when the interval Δ goes to zero. We call the limit proposals obtained in Theorem 2 the *weighted solidarity value*. This value (in the case of symmetric players) first appears in Sprumont (1990) as an example of a population monotonic allocation scheme. Later, and independently, Nowak and Radzik (1994) provide an alternative but equivalent formulation and give an axiomatic characterization of it. The limit proposals obtained in Theorem 3 are the *weighted Shapley value*. Shapley (1953) introduces his value in both the symmetric and asymmetric cases. Kalai and Samet (1987) offer a characterization of the family of weighted Shapley values which appears when the symmetry assumption is dropped.

In Section 4 we show how the payoffs of the players change when the parameters that specify these probabilities in the two bargaining models vary.

In Theorem 4 we consider the case of the weighted solidarity values. For any monotonic game we prove that: (1) being the proposer is always an advantage; (2) when the probability of being selected as a proposer increases, the expected payoffs increases; (3) if the game is also symmetric, when the probability of leaving the game after rejections increases the expected payoff decreases. This yields a clear link between the bargaining power of a player and these parameters. Increasing the probability of being the proposer increases bargaining power. On the other hand, the link with the probability of dropping out of the game is undetermined: It depends on the characteristic function v. Nevertheless, for the particular case in which players are indistinguishable (v is symmetric), if the probability of defeat increases bargaining power decreases.

In Theorem 5 we consider the case of the weighted Shapley values. The results obtained here are puzzling. Firstly, the effect of increasing the probability of being the proposer and of dropping out of the game always go in the same direction (in clear contrast with the solidarity case, where in symmetric games they go in opposite directions). However, there is no way of knowing whether they will be positive or negative: It depends on the characteristic function of the game, which seems a little arbitrary.

Only in the trivial case of pure bargaining games, where the grand coalition is the only productive one, do the two values coincide and the payoffs depend only on the relative probability of being the proposer.

Section 5 is devoted to the axiomatic approach. A usual method is to remove the property of symmetry between players from the set of axioms which characterizes the value, and to see what family of values emerges. However, we restrict ourselves to considering only two particular ways of differentiating players: by parameters α and ω. For that reason, we do not try to determine which whole family of Shapley and solidarity values could appear without symmetry. Instead, we seek to identify which values appear when the differences between players lie only in the differences in the probabilities specified by parameters α and ω. For that reason, those parameters are taken into account explicitly in the formulation of the axioms.

In Theorem 10 the weighted solidarity value is characterized via the properties of efficiency and *equal weighted average gains.* This last property says that the expected marginal contribution of society to each player's normalized payoff must be equal for them all. The parameter α determines the players' payoff normalization; and the parameter ω helps to compute the expected marginal contribution of society to each player.

In Theorem 12 the weighted Shapley value is characterized via the properties of efficiency and *weighted normalized balanced contributions.* This last property is a probabilistic interpretation of the balanced contributions property introduced in Myerson (1980) to characterize the *Shapley value*. It says that for each two players $i, j \in N$ the normalized (per-unit-weight) marginal contributions between pairs of players must be equal, where the parameter α determines the normalization of the payoffs and ω the probability of each player leaving the game. Hence, this property says that the value rewards players in such a way that each player contributes to the other the same as the other contributes to it.

Finally, in Section 6 we summarize our results and compare the two values from the positive/strategic and normative/axiomatic perspectives. From a normative point of view, evaluating whether it is more appropriate to reward players according to their productivity, as the Shapley value does, or in an egalitarian way, as the solidarity value does, depends on the context. However, from a strategic point of view, the consideration of asymmetries between players helps us to reach a better evaluation of the bargaining models under consideration. The monotonicity analysis of the payoffs with respect to the probabilities of players highlights behavioral aspects of these values which are more or less in line with our intuitions as to what happens in real negotiations. The main conclusion is that the bargaining protocol associated with the solidarity value seems more realistic than the one associated with the Shapley value. This shows the relevance of considering the case of asymmetric players in analyzing a value solution.

2 Bargaining

A *cooperative game* with transferable utility (TU-game) is a pair (N, v) where $N \subsetneq \mathbb{N}$[3] is a nonempty, finite set and $v : 2^N \to \mathbb{R}$ is a *characteristic function*, satisfying $v(\emptyset) = 0$. An element i of N is called a *player* and each nonempty subset S of N a *coalition*. The real number $v(S)$ is called the *worth* of coalition S, and is interpreted as the total payoff that the coalition S, if it forms, can obtain for its members. An example of this such games is a set of risk-neutral players who use a totally divisible good to make coalitional payoffs between them. Let G^N denote the set of all cooperative TU-games with player set N and let G denote the set of all games, that is, $G = \cup_{\emptyset \neq N \subsetneq \mathbb{N}} G^N$.

For all $S \subseteq N$, denote the restriction of (N, v) to S as (S, v). For the sake of simplicity, we write $S \cup i$ instead of $S \cup \{i\}$, $N \setminus i$ instead of $N \setminus \{i\}$, and $v(i)$ instead of $v(\{i\})$. For each vector $x \in \mathbb{R}^N$, let $x(S) := \sum_{i \in S} x_i$ for each $S \subseteq N$.

A TU-game is said to be *monotonic* if $v(T) \leq v(S)$ whenever $T \subseteq S$. In our setting it is explicitly assumed that the utilities are previously normalized in such a way that when any player leaves the game its payoff is *zero*. Monotonicity implies that for any subcoalition S the players have an incentive to cooperate because every player can attain payoffs better than it will obtain by being alone. Note also that the payoff $v(i)$ is what player i obtains if the remaining $N \backslash i$ players have left the game, so there is no need that $v(i) = 0$. For example, consider a bankruptcy situation where the liquidation value of a good must be divided between two creditors, i and j, each of which claims the whole value. The possible outcomes are either that the good is owned by only one of the players, say i obtains its claim and j receives nothing (i.e. $v(i) = 1$ and j leaves the game receiving zero) or the good can be shared by the two, i.e. $v(\{i, j\}) = 1$. This is a monotonic game in which players cannot guarantee their $v(i)$. If examples are to be considering, as market games, where players can guarantee their initial endowments without the help of the remaining players it suffices need to normalize the utilities such that $v(i) = 0$.

Moreover, we assume that players have *von Neumann-Morgenstern preferences* and are *risk-neutral*. Therefore, the expected utility of the lottery $\rho x \oplus (1 - \rho)\, y$ is $u_i\left(\rho x \oplus (1 - \rho)\, y\right) = \rho x + (1 - \rho)\, y$, where the probability of x is ρ, and the probability of y is $(1 - \rho)$.

A *value* is a function γ which assigns to each TU-game (N, v) and each player $i \in N$, a real number $\gamma_i(N, v)$, which represents an assessment made by i of its gains from participating in the game. A *payoff*

[3] $\mathbb{N}$ is the set of natural numbers.

configuration is an element of $\prod_{S\subseteq N}\mathbb{R}^S$.

We model the process that players must follow to find a cooperative agreement by an *alternating random proposer* protocol. This is a sequential, noncooperative game where the proposer is chosen at random at each step and players drop out of the game randomly after proposals are rejected:

> Consider a TU-game (N,v). In each round, there is a set $S\subseteq N$ of "active" players and a "proposer", who is chosen randomly from among them with a probability distribution $\mathbf{p}=\left(p^S\right)_{S\subseteq N}\left[p_i^S\geq 0\text{, for each } i\in S;\ \sum_{i\in S}p_i^S=1;\ \emptyset\neq S\subseteq N\right]$. In the first round all players are active, i.e. $S=N$. The proposer i makes a feasible offer $a^{S,i}\in\mathbb{R}^S$, i.e. $\sum_{j\in S}a_j^{S,i}\leq v(S)$. If the rest of the players accept the proposal the process ends with this offer. If it is rejected by even one player, the players move on to the next round where a coalition $T\subseteq S$ is chosen randomly to be the new set of active players with a probability distribution $\mathbf{P}^{S(i)}$ $\left[P_T^{S(i)}\geq 0\text{, for each } T\subseteq S;\ \sum_{T\subseteq S}P_T^{S(i)}=1;\ P_S^{S(i)}<1;\ S\subseteq N\right]$, and all players outside T (i.e., $j\in S\backslash T$) leave the game and receive a payoff of zero. Note that this probability distribution depends in general on who the proposer was, and it could happen that $\mathbf{P}^{S(i)}\neq\mathbf{P}^{S(j)}$ for each pair $i,j\in S$.[4]

Remark When $P_\emptyset^S=1$ for all $S\subseteq N$ this is an ultimatum game: the proposals are "take it or leave it" offers.

Remark This alternating random proposer protocol was introduced in Hart and Mas-Colell (1996).The version presented here is a little more general. There are two differences: 1) After each proposal is rejected, the probability of the new active set of bargaining players can depend on who the last proposer was; 2) In general, starting from an active coalition of size s, any subcoalition can be active and not only subcoalitions of size $s-1$.

Such bargaining process with three or more players has a broad range of subgame perfect equilibria associated with it. Hence, we follow the familiar route of considering only the *stationary* subgame perfect equilibria (in what follows SP equilibria).

Our first result characterizes the offers of an SP equilibrium.

Theorem 1 *Let (N,v) be a monotonic TU-game. Then, for each specification of the probability distributions $(\mathbf{p},\mathbf{P})$ there is an SP equilibrium. The proposals corresponding to an SP equilibrium are always accepted, and they are characterized by:*

(P.1) $a_i^{S,i}=v(S)-\sum_{j\in S\backslash i}a_j^{S,i}$ *for each $i\in S\subseteq N$; and*

(P.2) $a_j^{S,i}=\sum_{\substack{T\subseteq S\\ T\ni j}}P_T^{S(i)}a_j^T$ *for each $i,j\in S$ with $i\neq j$, and each $S\subseteq N$;*

where $a^S=\sum_{i\in S}p_i^S a^{S,i}$. Moreover, these proposals are unique and nonnegative.

[4] Although the vectors a^S and $a^{S,i}$ are functions of the probability distributions $\left(p_i^S\right)_{i\in S}$ and $\left(P_T^{S(i)}\right)_{\substack{i\in S\\ T\subseteq S}}$, we write a^S and $a^{S,i}$ instead of $a^S\left(\left(p_i^S\right)_{i\in S},\left(P_T^{S(i)}\right)_{\substack{i\in S\\ T\subseteq S}}\right)$ and $a^{S,i}\left(\left(p_i^S\right)_{i\in S},\left(P_T^{S(i)}\right)_{T\subseteq S}\right)$ for the sake of notational simplicity when no confusion can arise.

In other words, (P.2) says that i proposes to each $j \in S\backslash i$ the expected payoff that j would get in the continuation of the game in case of rejection, where $P_T^{S(i)}$ is the probability that coalition $T \subseteq S$ will be the new active set after the rejection of the proposal; and (P.1) says that i gets for itself the remainder up to the complete $v(S)$. Note that (P.1) and (P.2) imply efficiency of the proposals, i.e. $\sum_{j\in S} a_j^{S,i} = v(S)$, and hence the averages of the proposals are also efficient, i.e. $\sum_{j\in S} a_j^S = v(S)$.

In the proof we use the recursive formula in the following proposition.

Proposition 1 *Let $\left(a^S\right)_{S\subseteq N}$ be the average payoff configuration associated with proposals satisfying (P.1) and (P.2). Then, it holds that*

$$a_i^S = \frac{p_i^S}{1-\sum\limits_{j\in S} p_j^S P_S^{S(j)}}\left(v(S) - \sum_{T\subseteq S} P_T^{S(i)} v(T)\right) + \frac{1}{1-\sum\limits_{j\in S} p_j^S P_S^{S(j)}} \sum_{\substack{T\subsetneq S \\ T\ni i}} \left(\sum_{j\in S} p_j^S P_T^{S(j)}\right) a_i^T, \quad (i \in S \subseteq N), \tag{1}$$

Moreover, these vectors a^S, $S \subseteq N$, are unique and nonnegative.

Proof. Let (N, v) be a monotonic TU-game. By (P.1) , for any $i \in S \subseteq N$ we have

$$a_i^S = p_i^S\left(v(S) - \sum_{j\in S\backslash i} a_j^{S,i}\right) + \sum_{j\in S\backslash i} p_j^S a_i^{S,j}.$$

Applying (P.2),

$$\begin{aligned}
a_i^S &= p_i^S\left(v(S) - \sum_{j\in S\backslash i}\sum_{\substack{T\subseteq S \\ T\ni j}} P_T^{S(i)} a_j^T - \sum_{\substack{T\subseteq S \\ T\ni i}} P_T^{S(i)} a_i^T\right) + \sum_{j\in S\backslash i} p_j^S \sum_{\substack{T\subseteq S \\ T\ni i}} P_T^{S(j)} a_i^T + p_i^S \sum_{\substack{T\subseteq S \\ T\ni i}} P_T^{S(i)} a_i^T \\
&= p_i^S\left(v(S) - \sum_{T\subseteq S} P_T^{S(i)} \sum_{j\in S} a_j^T\right) + \sum_{\substack{T\subseteq S \\ T\ni i}}\left(\sum_{j\in S} p_j^S P_T^{S(j)}\right) a_i^T \\
&= p_i^S\left(v(S) - \sum_{T\subseteq S} P_T^{S(i)} \sum_{j\in T} a_j^T\right) + \sum_{\substack{T\subsetneq S \\ T\ni i}}\left(\sum_{j\in S} p_j^S P_T^{S(j)}\right) a_i^T + \left(\sum_{j\in S} p_j^S P_S^{S(j)}\right) a_i^S.
\end{aligned}$$

With efficiency, we have that

$$\begin{aligned}
a_i^S &= p_i^S\left(v(S) - \sum_{T\subseteq S} P_T^{S(i)} v(T)\right) + \sum_{\substack{T\subsetneq S \\ T\ni i}}\left(\sum_{j\in T} p_j^S P_T^{S(j)}\right) a_i^T + \left(\sum_{j\in S} p_j^S P_S^{S(j)}\right) a_i^S, \\
\left(1-\sum_{j\in S} p_j^S P_S^{S(j)}\right) a_i^S &= p_i^S\left(v(S) - \sum_{T\subseteq S} P_T^{S(i)} v(T)\right) + \sum_{\substack{T\subsetneq S \\ T\ni i}}\left(\sum_{j\in S} p_j^S P_T^{S(j)}\right) a_i^T, \\
a_i^S &= \frac{p_i^S}{1-\sum\limits_{j\in S} p_j^S P_S^{S(j)}}\left(v(S) - \sum_{T\subseteq S} P_T^{S(i)} v(T)\right) + \frac{1}{1-\sum\limits_{j\in S} p_j^S P_S^{S(j)}} \sum_{\substack{T\subsetneq S \\ T\ni i}}\left(\sum_{j\in S} p_j^S P_T^{S(j)}\right) a_i^T.
\end{aligned}$$

The payoffs of the single coalitions $\{i\}$, are $a_i^{\{i\}} = v(i)$, for all $i \in N$. By the induction hypothesis, it is known that $a_i^T \geq 0$ for all $i \in T \subsetneq S$, and by monotonicity,

$$v(S) \geq \sum_{T\subseteq S} P_T^{S(i)} v(T).$$

On the other hand, $\sum_{j\in S} p_j^S = 1$ and $P_S^{S(j)} < 1$ for all $j \in S$; thus $1 - \sum_{j\in S} p_j^S P_S^{S(j)} > 0$. This implies that $a_i^S \geq 0$ for all $i \in S$. ■

Proof of Theorem 1. The proof is provided by induction. It is immediate for the 1-player case. Assume that it holds when there are less than n players. Let $a^{S,i}$, for $i \in S \subseteq N$, be the proposals of a given SP-equilibrium. Denote by c^S the expected payoff vector for the members of S in the subgame where the set of active players is S. The induction hypothesis implies that $c^S = a^S$ and (P.1), (P.2) are satisfied for $S \neq N$.

Let $d^{N,i}$ be an SP equilibrium proposal vector of player $i \in N$. Among the offers that are accepted, the best proposal for i is to offer each other player $j \in N\backslash i$ its expected payoff in the case of rejection, namely

$$d_j^{N,i} = P_N^{N(i)} c_j^N + \sum_{\substack{T\subsetneq N \\ T\ni j}} P_T^{N(i)} a_j^T \ ,$$

and then take the entire surplus for itself:

$$d_i^{N,i} = v(N) - P_N^{N(i)} \left(\sum_{j\in N\backslash i} c_j^N \right) - \sum_{j\in N\backslash i} \sum_{\substack{T\subsetneq N \\ T\ni j}} P_T^{N(i)} a_j^T \ .$$

This is the proposal that is best for i out of those that will be accepted if i is the proposer. Moreover,

$$\begin{aligned}
d_i^{N,i} &= v(N) - P_N^{N(i)} \left(\sum_{j\in N} c_j^N \right) - \sum_{j\in N} \sum_{\substack{T\subsetneq N \\ T\ni j}} P_T^{N(i)} a_j^T + P_N^{N(i)} c_i^N + \sum_{\substack{T\subsetneq N \\ T\ni i}} P_T^{N(i)} a_i^T \\
&= v(N) - P_N^{N(i)} v(N) - \sum_{T\subsetneq N} P_T^{N(i)} \left(\sum_{j\in T} a_j^T \right) + P_N^{N(i)} c_i^N + \sum_{\substack{T\subsetneq N \\ T\ni i}} P_T^{N(i)} a_i^T \\
&= v(N) - P_N^{N(i)} v(N) - \sum_{T\subsetneq N} P_T^{N(i)} v(T) + P_N^{N(i)} c_i^N + \sum_{\substack{T\subsetneq N \\ T\ni i}} P_T^{N(i)} a_i^T \\
&= v(N) - \sum_{T\subseteq N} P_T^{N(i)} v(T) + P_N^{N(i)} c_i^N + \sum_{\substack{T\subsetneq N \\ T\ni i}} P_T^{N(i)} a_i^T \ .
\end{aligned}$$

By monotonicity,

$$v(N) - \sum_{T\subseteq N} P_T^{N(i)} v(T) \geq 0 \ ,$$

hence

$$d_i^{N,i} \geq P_N^{N(i)} c_i^N + \sum_{\substack{T\subsetneq N \\ T\ni i}} P_T^{N(i)} a_i^T,$$

which is what i expects to obtain in the continuation of the game in case of rejection.

On the other hand, any proposal of i that is rejected gives i at most $P_N^{N(i)} c_i^N + \sum_{\substack{T\subsetneq N \\ T\ni i}} P_T^{N(i)} a_i^T$, which is not greater than $d_i^{N,i}$. Hence, player i proposes $d^{N,i} = a^{N,i}$ and the proposal will be accepted. From this, it follows that $c^N = a^N$.

Nonnegativity still must be shown. First, note that the following strategy guarantees i a nonnegative payoff: as a respondent accept only a payoff of zero, and as a proposer offer zero for every $j \neq i$. This result implies that $a^{N,i}$ must be nonnegative.

Conversely, it can be shown that the proposals $\left(a_i^{S,i}\right)_{S\subset N, i\in S}$ satisfying (P.1) and (P.2) do form an SP equilibrium.

By Proposition 1 these proposals are unique and nonnegative. In the one-player case it is trivial that $a_i^{\{i\},i} = v(i)$ is an SP equilibrium. Assume that they form an SP equilibrium in any subgame with player set $S \neq N$. Fix a player $i \in N$. Given the strategies of the other players, as a proposer i cannot increase its payoff $a_i^{N,i}$ from proposals that are accepted. Making proposals that are systematically rejected it can only give the chance to go on to the breakdown stage, which provides it an expected payoff of $\sum_{\substack{T\subseteq N\\ T\ni i}} P_T^{N(i)} a_i^T$. The suggested strategies, however, yield $a_i^{N,i}$ which is better than $\sum_{\substack{T\subseteq N\\ T\ni i}} P_T^{N(i)} a_i^T$. As a respondent, i can only deviate by rejecting the offer $a_i^{N,j}$ made by another player j, but its expectation in the event of continuation, $\sum_{\substack{T\subseteq N\\ T\ni i}} P_T^{N(j)} a_i^T$, is just $a_i^{N,j}$. The only remaining option is to follow a strategy of dropping out of the game, but in that case the payoff is zero, whereas the payoffs associated with the proposed strategies in the SP equilibrium are nonnegative. ■

Remark Note that (P.1) implies that $a^{\{i\},i} = v(i)$, so $a^{\{i\},i}$ is nonrandom. Iterating in (P.2), this means that $a^{S,i}$ is also nonrandom. Therefore, in any stationary subgame perfect equilibrium mixed strategies can only appear when $a_i^{S,i} = a_i^{S,j}$. But in this case, as the proposer, player i can also claim an amount $b_i^{S,i} > a_i^{S,i}$ for itself. But this implies an amount $b_j^{S,i} < a_j^{S,i}$ for some $j \in S\backslash i$, so this proposal $b^{S,i}$ will be rejected by j for sure. In this case, the expected payoff associated with this strategy is again $a_i^{S,i} = a_i^{S,j}$. Therefore, for a proposer any mixed strategy between offering $a^{S,i}$ or $b^{S,i}$ always yields the same payoffs.

3 Strategic approach

3.1 Weighted solidarity values

When a cooperative solution is considered from an axiomatic point of view, asymmetric versions of the value appear when the property of symmetry[5] is removed from the set of axioms which characterizes the value. The type of reason that justifies such a lack of symmetry depends on the context. It could be justified by differences in the negotiating skills of players (whatever that means). They may be representatives of groups of different sizes, or there may be differences in the probabilities of staying in the game when time passes with no agreement, etc. However, in the strategic approach, the noncooperative game that models the bargaining process must be completely specified. Hence, the source of asymmetric payoffs must necessarily be associated with a particular specification of the parameters in the game. This helps to understand why and to what extent such asymmetries will arise in the payoffs.

Consider the general family of values corresponding to the case where, given two fixed vectors $\alpha, \omega \in \mathbb{R}^N$ with $\alpha_i > 0$ and $\omega_i > 0$ for each $i \in N$, and a parameter $\rho \in \mathbb{R}$ with $0 \leq \rho < 1$, for all $S \subseteq N$ we

[5] Two players $i, j \in N$ are *symmetric* in (N, v) if $v(S \cup i) = v(S \cup j)$ for all $S \subseteq N \setminus \{i, j\}$. A value γ satisfies *symmetry* if $\gamma_i(N, v) = \gamma_j(N, v)$ whenever i and j are symmetric.

define:

$$\begin{aligned} p_i^S &= \frac{\alpha_i}{\alpha(S)}, \qquad (i \in S) \ , \\ P_T^{S(i)} &= \prod_{j \in T} \rho^{\omega_j} \prod_{j \in S \setminus T} (1 - \rho^{\omega_j}) \, , \quad (T \subseteq S) \ . \end{aligned}$$

Here, each player $i \in S$ has its own (independent) probability $p_i^S = \frac{\alpha_i}{\alpha(S)}$ of being selected as a proposer, and after rejection each player has its own (independent) probability ρ^{ω_i} of remaining as an active player. Increasing α_i increases the probability of being selected as a proposer, and increasing ω_i decreases the probability ρ^{ω_i} of continuing in the game after rejection. Given the independence assumption, the probability of coalition $T \subseteq S$ being a new active coalition is $P_T^{S(i)} = \prod_{j \in T} \rho^{\omega_j} \prod_{j \in S \setminus T} (1 - \rho^{\omega_j})$, for each $T \subseteq S$. It is clear here that this probability is independent of who was the proposer, i.e. $P_T^{S(i)} = P_T^{S(j)}$ for all $i, j \in S$.

One interpretation of these parameters is the following.

The differences in α_i fits very well when players are representatives of groups of different size. For example, consider the problem of distributing profits among the teams/departments of a firm. Any amount of money given to a team can be freely distributed among its members. Assume that the team can do its work only when it is complete, as all of its members are equally necessary for its completeness, which means that they are symmetric players. Moreover, as individual members (when the team is not complete), each player alone does not contribute to the productivity of the other teams. So they must be taken into account only jointly with the rest of the team in the distribution of profits. This gives an economic justification for replacing the original game of workers by the game of teams. Suppose that in the bargaining process all workers have the same probability of being selected as a proposer in an active coalition S of completed teams. This implies that the probability of selecting a representative of a team i is $\frac{\alpha_i}{\alpha(S)}$, where α_i measures the number of workers in team i.

Other examples include simple voting games in which players are parties with different numbers of seats in a parliament; sharing the allocation of the costs of a public facility among cities or communities of different sizes; international agreements between countries with different population sizes, and so on.

Alternatively, it can be assumed that players have different knowledge about the data of the problem so some can build proposals faster than others, with α_i being the parameter which measures that ability.

One interpretation of ρ^{ω_i} is based in the fact that negotiations take place at time $\lambda \in [0, \infty)$. Each player i has its own probability $p_i(\lambda)$ of being in the game up to time λ. Suppose that the rate at which this probability changes is proportional, i.e. $dp_i(\lambda) = -\omega_i p_i$. Here, ω_i is a positive, constant coefficient of proportionality, fixed by characteristics of the player. One possibility is a fitness characteristic such as expected time of life or vitality. A negative sign means that $p_i(\lambda)$ is decreasing over time. Taking the initial condition $p_i(0) = 1$, the solution of this ordinal differential equation is $p_i(\lambda) = e^{-\omega_i \lambda}$. The sequence of rounds can be enumerated by $t = 0, 1, 2, ...$, with $\Delta > 0$ denoting the time that each round takes. It is now possible to write $p_i(t) = e^{-\omega_i \Delta t}$ as the probability of player i being in the game at round t. If $\rho = e^{-\Delta}$ then $p_i(t) = \rho^{\omega_i t}$, and under the stationary assumption this means that, after a rejection the probability of being in the game at round t, conditioned to still being in the game at round $t-1$, is

ρ^{ω_i}. When the time taken by each round Δ approaches zero, ρ converges to 1, and so $\rho^{\omega_i} \to 1$. Note that when $\rho \to 1$ the rates of convergence between players are constant, measured in proportional terms. That is, the cross elasticity $\mathcal{E}(\rho^{\omega_i}, \rho^{\omega_j})$ is constant:

$$\mathcal{E}(\rho_i, \rho_j) = \frac{\partial \ln \rho^{\omega_i}}{\partial \ln \rho^{\omega_j}} = \frac{\omega_i}{\omega_j}, \quad (i, j \in N) \ .$$

We now show an explicit formula for the average proposals obtained when $\rho \to 1$.

First some definitions are needed. Let (N, v) be a TU-game. For each coalition $S \subseteq N$ and player $i \in S$, the term $\Delta^i(v, S) := v(S) - v(S \backslash i)$ is the *marginal contribution of player i to coalition S in the TU-game* (N, v).

For each coalition $S \subseteq N$, define

$$\Delta^{av}_{\omega}(v, S) := \sum_{i \in S} \frac{\omega_i}{\omega(S)} \Delta^i(v, S) \ .$$

The expression $\Delta^{av}_{\omega}(v, S)$ is the *weighted average of the marginal contributions of players within coalition S in the game* (N, v).

Given vectors $\alpha, \omega \in \mathbb{R}^N_{++}$, for notational convenience we define $\alpha\omega(S) = \sum_{i \in S} \alpha_i \omega_i$, for all $S \subseteq N$.

The *weighted solidarity value* $Sl^{\alpha\omega}$ is defined inductively by

$$Sl^{\alpha\omega}_i(S, v) = \frac{\alpha_i}{\alpha(S)} \Delta^{av}_{\omega}(v, S) + \sum_{j \in S \backslash i} \frac{\omega_j}{\omega(S)} Sl^{\alpha\omega}_i(S \backslash j, v), \quad (i \in S \subseteq N) \ , \tag{2}$$

starting with

$$Sl^{\alpha\omega}_i(\{i\}, v) = v(i), \quad (i \in N) \ .$$

Remark Formula (2) clearly shows the homogeneity of payoffs with respect to α and ω. Therefore, payoffs are only sensitive to changes in the relative weights.

Remark The family of values $Sl^{\alpha\omega}$ introduced here has some precedents in the literature.

Sprumont (1990; Section 5) introduces the value Sl defined recursively by

$$Sl_i(S, v) = \frac{1}{s} \Delta^{av}(v, S) + \sum_{j \in S \backslash i} \frac{1}{s} Sl_i(S \backslash j, v), \quad (i \in S \subseteq N) \ , \tag{3}$$

starting with

$$Sl_i(\{i\}, v) = v(i), \quad (i \in N) \ .$$

It is clear that $Sl \equiv \varphi^{\alpha\omega}$ when $\alpha_i = \alpha_j$ and $\omega_i = \omega_j$ for all $i, j \in N$.

The next formula

$$Sl_i(N, v) = \sum_{\substack{S \subseteq N \\ i \epsilon S}} \frac{(n - s)! \, (s - 1)!}{n!} \Delta^{av}(v, S), \quad (i \in N) \ , \tag{4}$$

where

$$\Delta^{av}(v, S) = \frac{1}{s} \sum_{i \in S} \Delta^i(v, S) \ ,$$

was introduced by Nowak and Radzik (1994) to define what they called the *Solidarity value* of the game (N, v). They note that this value does not satisfy the null player axiom. A player i in the game (N, v)

is called a *null player* if $\Delta^i(v,S) = 0$ for each coalition $S \subseteq N$ containing i. A value γ satisfies the *null player axiom* if $\boldsymbol{\gamma}_i(N,v) = 0$ when i is a null player in (N,v). Instead, they propose an average null player axiom. We say that a player i in the game (N,v) is an *average null player* if $\Delta^{av}(v,S) = 0$ for each coalition $S \subseteq N$ containing i. A value γ satisfies the *average null player axiom* if $\boldsymbol{\gamma}_i(N,v) = 0$ when i is an average null player in (N,v). They offer an axiomatic characterization of Sl parallel to the characterization of the *Shapley value* (Shapley, 1953b), replacing the null player axiom by the average null player axiom.

Calvo (2008) shows that definitions (3) and (4) are equivalent. Calvo and Gutiérrez-López (2014) consider the subfamily of values $Sl^{\alpha} \equiv Sl^{\alpha\omega}$ with $\omega_i = \omega_j$ for all $i,j \in N$; and give an axiomatic characterization for that subfamily.

We now show an explicit formula for the SP equilibrium proposals of Theorem 1 when the interval of time between offers tends to zero: $\Delta \to 0$, which implies that $\rho \to 1$.

Theorem 2 *Let (N,v) be a monotonic TU-game, and parameters $\alpha, \omega \in \mathbb{R}^N_{++}$. Then, for every coalition $S \subseteq N$, and players $i,j \in S$, it holds that*
(1) $a_i^{S,i} \geq a_i^{S,j}$,
(2) $\lim_{\rho\to 1}\left(a_i^{S,i}(\rho) - a_i^{S,j}(\rho)\right) = 0$,
(3) $\lim_{\rho\to 1} a_i^S(\rho) = Sl_i^{\alpha\omega}(S,v)$.

Proof. Let $S \subseteq N$. First denote $P_T^S = \prod_{i\in T} \rho^{\omega_i} \prod_{j\in S\setminus T} (1-\rho^{\omega_j})$, for all $T \subseteq S$.
(1) Let players $i,j \in S$, then

$$\begin{aligned} a_i^{S,i} - a_i^{S,j} &= v(S) - \sum_{j\in S\setminus i} a_j^{S,i} - \sum_{\substack{T\subseteq S \\ T\ni i}} P_T^{S(j)} a_i^T = v(S) - \sum_{j\in S\setminus i}\left[\sum_{\substack{T\subseteq S \\ T\ni j}} P_T^{S(i)} a_j^T\right] - \sum_{\substack{T\subseteq S \\ T\ni i}} P_T^{S(j)} a_i^T \\ &= v(S) - \sum_{T\subseteq S} P_T^S \left(\sum_{j\in T} a_j^T\right) = v(S) - \sum_{T\subseteq S} P_T^S v(T) \geq 0 \ , \end{aligned}$$

because $\rho^r < 1$, $\sum_{T\subseteq S} P_T^S = 1$, and, by monotonicity, $v(T) \leq v(S)$ for all $T \subseteq S$.
(2) Let players $i,j \in S$, then

$$\begin{aligned} \lim_{\rho\to 1}\left(a_i^{S,i}(\rho) - a_i^{S,j}(\rho)\right) &= \lim_{\rho\to 1}\left(v(S) - \sum_{T\subseteq S} \rho^{\omega(T)} \prod_{j\in S\setminus T} (1-\rho^{\omega_j})\, v(T)\right) \\ &= v(S) - \lim_{\rho\to 1} \rho^{\omega(S)} v(S) - - \lim_{\rho\to 1}\left(\sum_{T\subsetneq S} \rho^{\omega(T)} \prod_{j\in S\setminus T}(1-\rho^{\omega_j})\, v(T)\right) \\ &= v(S) - v(S) - 0 = 0 \ . \end{aligned}$$

(3) The proof is provided by induction. When $S = \{i\}$, it holds that

$$a_i^{\{i\}}(\rho) = v(i) = Sl_i^{\alpha\omega}(\{i\}, v) \ .$$

Assume that $a_i^T(\rho) \to Sl_i^{\alpha\omega}(T,v)$ for each $T \subsetneq S$ and each $i \in T$. For each $i \in S$, following formula (1)

$$\begin{aligned} a_i^S(\rho) &= \frac{\frac{\alpha_i}{\alpha(S)}}{1-\sum\limits_{j\in S}\frac{\alpha_j}{\alpha(S)}P_S^{S(j)}}\left(v(S)-\sum_{T\subseteq S}P_T^{S(i)}v(T)\right)+\frac{1}{1-\sum\limits_{j\in S}\frac{\alpha_j}{\alpha(S)}P_S^{S(j)}}\sum_{\substack{T\subsetneq S\\T\ni i}}\left(\sum_{j\in S}\frac{\alpha_j}{\alpha(S)}P_T^{S(j)}\right)a_i^T(\rho) \\ &= \frac{\frac{\alpha_i}{\alpha(S)}}{1-\rho^{\omega(S)}}\left(v(S)-\rho^{\omega(S)}v(S)-\sum_{T\subsetneq S}\rho^{\omega(T)}\prod_{j\in S\setminus T}(1-\rho^{\omega_j})\,v(T)\right) \\ &\quad +\frac{1}{1-\rho^{\omega(S)}}\sum_{\substack{T\subsetneq S\\T\ni i}}\rho^{\omega(T)}\prod_{j\in S\setminus T}(1-\rho^{\omega_j})\,a_i^T(\rho)\ , \end{aligned}$$

and then

$$a_i^S(\rho)=\frac{\alpha_i}{\alpha(S)}\left(v(S)-\sum_{T\subsetneq S}\frac{\rho^{\omega(T)}\prod\limits_{j\in S\setminus T}(1-\rho^{\omega_j})}{1-\rho^{\omega(S)}}v(T)\right)+\sum_{\substack{T\subsetneq S\\T\ni i}}\frac{\rho^{\omega(T)}\prod\limits_{j\in S\setminus T}(1-\rho^{\omega_j})}{1-\rho^{\omega(S)}}a_i^T(\rho)\ . \quad (5)$$

Applying l'Hôpital's rule, when $\rho \to 1$,

$$\lim_{\rho\to1}\frac{\rho^{\omega(T)}\prod\limits_{j\in S\setminus T}(1-\rho^{\omega_j})}{1-\rho^{\omega(S)}}=\lim_{\rho\to1}\frac{\omega(T)\rho^{\omega(T)-1}\prod\limits_{j\in S\setminus T}(1-\rho^{\omega_j})-\rho^{\omega(T)}\sum\limits_{j\in S\setminus T}\omega_j\rho^{\omega_j-1}\prod\limits_{k\in(S\setminus T)\setminus j}(1-\rho^{\omega_k})}{-\omega(S)\rho^{\omega(S)-1}}\ .$$

When $|T| = |S| - 1$ it holds that $S\setminus T = \{j\}$, $j \in S$. Hence,

$$\begin{aligned} &\lim_{\rho\to1}\frac{\omega(T)\rho^{\omega(T)-1}\prod\limits_{j\in S\setminus T}(1-\rho^{\omega_j})-\rho^{\omega(T)}\sum\limits_{j\in S\setminus T}\omega_j\rho^{\omega_j-1}\prod\limits_{k\in(S\setminus T)\setminus j}(1-\rho^{\omega_k})}{-\omega(S)\rho^{\omega(S)-1}} \\ &\qquad =\lim_{\rho\to1}\frac{\omega(T)\rho^{\omega(T)-1}(1-\rho^{\omega_j})-\rho^{\omega(T)}\omega_j\rho^{\omega_j-1}}{-\omega(S)\rho^{\omega(S)-1}}=\frac{\omega_j}{\omega(S)}\ , \end{aligned}$$

and for all $|T| < |S| - 1$ it holds that $|S\setminus T| \geq 2$, then

$$\lim_{\rho\to1}\frac{\omega(T)\rho^{\omega(T)-1}\prod\limits_{j\in S\setminus T}(1-\rho^{\omega_j})-\rho^{\omega(T)}\sum\limits_{j\in S\setminus T}\omega_j\rho^{\omega_j-1}\prod\limits_{k\in(S\setminus T)\setminus j}(1-\rho^{\omega_k})}{-\omega(S)\rho^{\omega(S)-1}}=\frac{0}{-\omega(S)}=0\ .$$

Hence, applying the induction hypothesis,

$$\begin{aligned} \lim_{\rho\to1}a_i^S(\rho) &= \lim_{\rho\to1}\left[\frac{\alpha_i}{\alpha(S)}\left(v(S)-\sum_{T\subsetneq S}\frac{\rho^{\omega(T)}\prod\limits_{j\in S\setminus T}(1-\rho^{\omega_j})}{1-\rho^{\omega(S)}}v(T)\right)+\sum_{\substack{T\subsetneq S\\T\ni i}}\frac{\rho^{\omega(T)}\prod\limits_{j\in S\setminus T}(1-\rho^{\omega_j})}{1-\rho^{\omega(S)}}a_i^T(\rho).\right] \\ &= \frac{\alpha_i}{\alpha(S)}\left(v(S)-\sum_{j\in S}\frac{\omega_j}{\omega(S)}v(S\setminus j)\right)+\sum_{j\in S\setminus i}\frac{\omega_j}{\omega(S)}Sl_i^{\alpha\omega}(S\setminus j,v) \\ &= \frac{\alpha_i}{\alpha(S)}\left(\sum_{j\in S}\frac{\omega_j}{\omega(S)}\left(v(S)-v(S\setminus j)\right)\right)+\sum_{j\in S\setminus i}\frac{\omega_j}{\omega(S)}Sl_i^{\alpha\omega}(S\setminus j,v) \\ &= \frac{\alpha_i}{\alpha(S)}\Delta_\omega^{av}(v,S)+\sum_{j\in S\setminus i}\frac{\omega_j}{\omega(S)}Sl_i^{\alpha\omega}(S\setminus j,v)=Sl_i^{\alpha\omega}(S,v)\ . \end{aligned}$$

■

That is: (1) implies that being the proposer is always an advantage; (2) means that this advantage disappears as the bargaining time interval Δ decreases, as all equilibrium proposals converge with one another, and hence converge to the average; (3) says that these average proposals converge to the weighted solidarity value $Sl^{\alpha\omega}$.

3.2 Weighted Shapley values

We now introduce the two types of asymmetry considered for the weighted solidarity values in the context of the weighted Shapley values. This makes it easier to compare the two values.

For each $\emptyset \neq S \subseteq N$, the unanimity game (N, u_S) is defined as $u_S(T) = 1$ if $S \subseteq T$, and $u_S(T) = 0$ otherwise. Let $w \in \mathbb{R}^N$ be a positive weight vector. The *weighted Shapley value* Sh^w (Shapley 1953a) is the linear mapping defined for each unanimity game (N, u_S), $\emptyset \neq S \subseteq N$, as follows

$$Sh_i^w(N, u_S) = \begin{cases} \frac{w_i}{w(S)} & \text{if } i \in S \ , \\ 0 & \text{otherwise} \ . \end{cases}$$

The value Sh^w can also be defined inductively by

$$Sh_i^w(S, v) = \frac{w_i}{w(S)} \left(v(S) - v(S \backslash i)\right) + \sum_{j \in S \backslash i} \frac{w_j}{w(S)} Sh_i^w(S \backslash j, v), \quad (i \in S \subseteq N) \ ,$$

starting with

$$Sh_i^w(\{i\}, v) = v(i), \quad (i \in N) \ .$$

In the bargaining game defined in Hart and Mas-Colell (1996), the probability of each player being the proposer is also $p_i^S = \frac{\alpha_i}{\alpha(S)}$ for all $i \in S \subseteq N$. The only difference with the solidarity setting appears in what happens when a proposal of player i is rejected. Now only the proposer i has a chance to withdraw from the game. This means that the probability distribution $\mathbf{P}^{S(i)}$ is given by: $P_S^{S(i)} = \rho^{\omega_i}$, $P_{S\backslash i}^{S(i)} = (1 - \rho^{\omega_i})$, and $P_T^{S(i)} = 0$ for all $T \neq S, S\backslash i$.

We show an explicit formula for the SP equilibrium proposals obtained when the time between offers tends to zero: $\Delta \to 0$, that is, $\rho \to 1$.

Theorem 3 *Let (N, v) be a monotonic TU-game, and parameters $\alpha, \omega \in \mathbb{R}^N_{++}$. Then, for every coalition $S \subseteq N$, and players $i, j \in S$, it holds:*
(1) $\lim_{\rho \to 1} \left| a_i^{S,i}(\rho) - a_i^{S,j}(\rho) \right| = 0$,
(2) $\lim_{\rho \to 1} a_i^S(\rho) = Sh_i^w(S, v)$, *where $w_i = \alpha_i \omega_i$ for all $i \in N$.*

Proof. Let $T \subseteq S \subseteq N$. First note that as $P_S^{S(i)} = \rho^{\omega_i}$, $P_{S\backslash i}^{S(i)} = \left(1 - \rho^{\omega i}\right)$, and $P_T^{S(i)} = 0$ otherwise, we have that

$$a_j^{S,i}(\rho) = \rho^{\omega_i} a_j^S(\rho) + (1 - \rho^{\omega_i}) a_j^{S \backslash i}(\rho) \quad (i, j \in S, \ i \neq j; \ S \subseteq N) \ .$$

(1) Let players $i, j \in S$, then

$$\begin{aligned} \lim_{\rho \to 1} \left| a_i^{S,i}(\rho) - a_i^{S,j}(\rho) \right| &= \lim_{\rho \to 1} \left| v(S) - \sum_{j \in S \backslash i} a_j^{S,i}(\rho) - a_i^{S,j}(\rho) \right| \\ &= \lim_{\rho \to 1} \left| v(S) - \sum_{j \in S \backslash i} \left(\rho^{\omega_i} a_j^S(\rho) + (1 - \rho^{\omega_i}) a_j^{S \backslash i}(\rho) \right) - \left(\rho^{\omega_j} a_i^S(\rho) + (1 - \rho^{\omega_j}) a_i^{S \backslash j}(\rho) \right) \right| \\ &= |v(S) - v(S)| = 0 \ . \end{aligned}$$

(2) The proof is provided by induction. When $S = \{i\}$, it holds that

$$a_i^{\{i\}}(\rho) = v(i) = Sh_i^{\alpha\omega}(\{i\}, v) \ .$$

Assume that $a^T(\rho) \to Sh^w(T, v)$ for each $T \subsetneq S$, with $w_i = \alpha_i \omega_i$ for all $i \in N$.

By (P.1), for any $i \in S \subseteq N$ we have

$$a_i^S(\rho) = \frac{\alpha_i}{\alpha(S)} \left(v(S) - \sum_{j \in S \setminus i} a_j^{S,i}(\rho) \right) + \sum_{j \in S \setminus i} \frac{\alpha_j}{\alpha(S)} a_i^{S,j}(\rho) \ .$$

Applying (P.2),

$$\begin{aligned}
\alpha(S)\, a_i^S(\rho) &= \alpha_i v(S) - \alpha_i \rho^{\omega_i} \sum_{j \in S \setminus i} a_j^S(\rho) - \alpha_i (1 - \rho^{\omega_i}) \sum_{j \in S \setminus i} a_j^{S \setminus i}(\rho) \\
&\quad + \sum_{j \in S \setminus i} \alpha_j \rho^{\omega_j} a_i^S(\rho) + \sum_{j \in S \setminus i} \alpha_j (1 - \rho^{\omega_j})\, a_i^{S \setminus j}(\rho) + \alpha_i \rho^{\omega_i} a_i^S(\rho) - \alpha_i \rho^{\omega_i} a_i^S(\rho) \\
&= \alpha_i v(S) - \alpha_i \rho^{\omega i} v(S) - \alpha_i (1 - \rho^{\omega_i})\, v(S \setminus i) \\
&\quad + \left(\sum_{j \in S} \alpha_j \rho^{\omega_j} \right) a_i^S(\rho) + \sum_{j \in S \setminus i} \alpha_j (1 - \rho^{\omega_j})\, a_i^{S \setminus j}(\rho) \ .
\end{aligned}$$

Hence

$$\left(\alpha(S) - \sum_{j \in S} \alpha_j \rho^{\omega_j} \right) a_i^S(\rho) = \alpha_i (1 - \rho^{\omega_i}) (v(S) - v(S \setminus i)) + \sum_{j \in S \setminus i} \alpha_j (1 - \rho^{\omega_j})\, a_i^{S \setminus j}(\rho) \ .$$

An then

$$a_i^S(\rho) = \frac{\alpha_i (1 - \rho^{\omega_i})}{\alpha(S) - \sum_{j \in S} \alpha_j \rho^{\omega_j}} (v(S) - v(S \setminus i)) + \sum_{j \in S \setminus i} \frac{\alpha_j (1 - \rho^{\omega_j})}{\alpha(S) - \sum_{j \in S} \alpha_j \rho^{\omega_j}} a_i^{S \setminus j}(\rho) \ . \qquad (6)$$

Applying l'Hôpital's rule, when $\rho \to 1$,

$$\lim_{\rho \to 1} \frac{\alpha_i (1 - \rho^{\omega_i})}{\alpha(S) - \sum_{j \in S} \alpha_j \rho^{\omega_j}} = \lim_{\rho \to 1} \frac{-\alpha_i \omega_i \rho^{\omega_i - 1}}{-\sum_{j \in S} \alpha_j \omega_j \rho^{\omega_j - 1}} = \frac{\alpha_i \omega_i}{\sum_{j \in S} \alpha_j \omega_j} \ .$$

Finally, applying the induction hypothesis,

$$\lim_{\rho \to 1} a_i^S(\rho) = \frac{\alpha_i \omega_i}{\sum_{j \in S} \alpha_j \omega_j} (v(S) - v(S \setminus i)) + \sum_{j \in S \setminus i} \frac{\alpha_j \omega_j}{\sum_{j \in S} \alpha_j \omega_j} Sh_i^{\alpha\omega}(S \setminus j, v) = Sh_i^{\alpha\omega}(S, v) \ .$$

■

4 Monotonicity properties of parameters α, ω

The introduction of asymmetries between players in the value $Sl^{\alpha\omega}$ is explicitly given by the differences in the parameters α, ω. In this section we analyze the variation in the values $Sl^{\alpha\omega}$ and $Sh^{\alpha\omega}$ due to the change in these parameters.

This leads to the following theorems.

Theorem 4 *Let (N,v) be a monotonic TU-game and parameters $\alpha,\omega \in \mathbb{R}^N_{++}$. Then, for each $i \in S \subseteq N$, it holds that*

$$\begin{aligned}
&(D.1)\ \frac{\partial}{\partial \alpha_i} Sl_i^{\alpha\omega}(S,v) &\geq& \ 0\ ,\\
&(D.2)\ \frac{\partial}{\partial \omega_i} Sl_i^{\alpha\omega}(S,v) &\gtrless& \ 0\ ,\\
&(D.3)\ \frac{\partial}{\partial \omega_i} Sl_i^{\alpha\omega}(S,v) &\leq& \ 0 \text{ when } v(S) = f(|S|)\,,\ (S \subseteq N)\ .
\end{aligned}$$

Proof. As $Sl_i^{\alpha\omega}(\{i\},v) = v(i)$, when $|S| = 1$, the proposition trivially holds. Assume by induction that the proposition is true for all $T \subsetneq S$. In order to prove D.1, note that

$$\begin{aligned}
\frac{\partial}{\partial \alpha_i} Sl_i^{\alpha\omega}(S,v) &= \frac{\partial}{\partial \alpha_i}\left[\frac{\alpha_i}{\alpha(S)}\Delta_\omega^{av}(v,S) + \sum_{j\in S\backslash i}\frac{\omega_j}{\omega(S)} Sl_i^{\alpha\omega}(S\backslash j,v)\right]\\
&= \frac{\alpha(S\backslash i)}{\alpha(S)^2}\Delta_\omega^{av}(v,S) + \sum_{j\in S\backslash i}\frac{\omega_j}{\omega(S)}\frac{\partial}{\partial \alpha_i} Sl_i^{\alpha\omega}(S\backslash j,v)\ .
\end{aligned}$$

As $\alpha(S\backslash i) > 0$, $\Delta_\omega^{av}(v,S) \geq 0$ by monotonicity, and $\frac{\partial}{\partial \alpha_i} Sl_i^{\alpha\omega}(S\backslash j,v) \geq 0$ by the induction hypothesis, then $\frac{\partial}{\partial \alpha_i} Sl_i^{\alpha\omega}(S,v) \geq 0$.

To see D.2, note that

$$\begin{aligned}
\frac{\partial}{\partial \omega_i} Sl_i^{\alpha\omega}(S,v) &= \frac{\partial}{\partial \omega_i}\left[\frac{\alpha_i}{\alpha(S)}\Delta_\omega^{av}(v,S) + \sum_{j\in S\backslash i}\frac{\omega_j}{\omega(S)} Sl_i^{\alpha\omega}(S\backslash j,v)\right]\\
&= \frac{\alpha_i}{\alpha(S)}\sum_{j\in S\backslash i}\frac{\omega_j\left(v(S\backslash j) - v(S\backslash i)\right)}{\omega(S)^2} - \sum_{j\in S\backslash i}\frac{\omega_j}{\omega(S)^2} Sl_i^{\alpha\omega}(S\backslash j,v) + \sum_{j\in S\backslash i}\frac{\omega_j}{\omega(S)}\frac{\partial}{\partial \omega_i} Sl_i^{\alpha\omega}(S\backslash j,v)\ .
\end{aligned}$$

In general, the sign of $\sum_{j\in S\backslash i}\omega_j\left(v(S\backslash j) - v(S\backslash i)\right)$ is not determined, hence the sign of $\frac{\partial}{\partial \omega_i} Sl_i^{\alpha\omega}(S,v)$ is also undetermined.

To see D.3, note that when $v(S) = f(|S|)$, for all $S \subseteq N$ it holds that $\sum_{j\in S\backslash i}\omega_j\left(v(S\backslash j) - v(S\backslash i)\right) = 0$. Moreover, for each $j \neq i$ it holds that $Sl_i^{\alpha\omega}(S\backslash j,v) \geq 0$ by definition (2), so applying induction if it is assumed that $\frac{\partial}{\partial \omega_i} Sl_i^{\alpha\omega}(S\backslash j,v) \leq 0$ for all $j \neq i$, we conclude that $\frac{\partial}{\partial \omega_i} Sl_i^{\alpha\omega}(S,v) \leq 0$. ■

For the interpretation of D.1, recall that $\frac{\alpha_i}{\alpha(S)}$ is the probability of player i being selected as the proposer. A parameter α_i greater than α_j means that i is a more active player than j in the bargaining process. This should be associated with a stronger bargaining position. Hence, with the remaining parameters fixed, raising α_i should mean raising the payoff $Sl_i^{\alpha\omega}$.

On the other hand, the probability of player i remaining in the game in round t, conditional on being in the game in round $t-1$, is ρ^{ω_i}, where ω_i is a parameter that reflects its vitality, or fitness. As $\rho < 1$, a greater value ω_i means a lower probability ρ^{ω_i} of remaining after each rejection. D.2 says that when this probability of leaving the game changes, the power bargaining position of the player can change in either direction, so it is undetermined: it depends on the characteristic function of the game.

Nevertheless, there is a particular case in which $\frac{\partial}{\partial \omega_i} Sl_i^{\alpha\omega}$ is fully determined. This is the case when the game (N,v) is *symmetric*, that is $v(S) = f(|S|)$, for all $S \subseteq N$. Therefore, all players are indistinguishable by the characteristic function v, hence they only differ according to parameters α and ω. In this case, D.3 says that when ω_i *rises* (the probability ρ^{ω_i} of continuing in the game decreases), $Sl_i^{\alpha\omega}$ *decreases*, and thus the bargaining power of player i decreases.

Now consider the weighted Shapley values. In our setting the components w_i of the weight vector are the products $\alpha_i\omega_i$, so we denote $Sh^{\alpha\omega} \equiv Sh^w$.

Theorem 5 *Let (N,v) be a monotonic TU-game and parameters $\alpha,\omega \in \mathbb{R}^N_{++}$. Then, for each $i \in S \subseteq N$, it holds that*

$$
\begin{aligned}
\text{(E.1)}\ \frac{\partial}{\partial \alpha_i} Sh_i^{\alpha\omega}(S,v) &\gtrless 0\ , \\
\text{(E.2)}\ \frac{\partial}{\partial \omega_i} Sh_i^{\alpha\omega}(S,v) &\gtrless 0\ , \\
\text{(E.3)}\ \frac{1}{\omega_i}\frac{\partial}{\partial \alpha_i} Sh_i^{\alpha\omega}(S,v) &= \frac{1}{\alpha_i}\frac{\partial}{\partial \omega_i} Sh_i^{\alpha\omega}(S,v)\ .
\end{aligned}
$$

Proof. Let (N,v) be a monotonic TU-game and player $i \in S \subseteq N$. Denote $(\alpha\omega)(S) = \sum_{i\in S}\alpha_i\omega_i$. Then

$$
\begin{aligned}
\frac{\partial}{\partial \alpha_i} Sh_i^{\alpha\omega}(S,v) &= \frac{\partial}{\partial \alpha_i}\left[\frac{\alpha_i\omega_i}{\alpha\omega(S)}\left(v(S)-v(S\backslash i)\right) + \sum_{j\in S\backslash i}\frac{\alpha_j\omega_j}{\alpha\omega(S)} Sh_i^{\alpha\omega}(S\backslash j,v)\right] \\
&= \frac{\omega_i\,(\alpha\omega)(S\backslash i)}{\alpha\omega(S)^2}\left(v(S)-v(S\backslash i)\right) - \sum_{j\in S\backslash i}\frac{\alpha_j\omega_j\omega_i}{\alpha\omega(S)^2} Sh_i^{\alpha\omega}(S\backslash j,v) \\
&\quad + \sum_{j\in S\backslash i}\frac{\alpha_j\omega_j}{\alpha\omega(S)}\frac{\partial}{\partial \alpha_i} Sh_i^{\alpha\omega}(S\backslash j,v)\ ,
\end{aligned}
$$

and

$$
\begin{aligned}
\frac{\partial}{\partial \omega_i} Sh_i^{\alpha\omega}(S,v) &= \frac{\partial}{\partial \omega_i}\left[\frac{\alpha_i\omega_i}{\alpha\omega(S)}\left(v(S)-v(S\backslash i)\right) + \sum_{j\in S\backslash i}\frac{\alpha_j\omega_j}{\alpha\omega(S)} Sh_i^{\alpha\omega}(S\backslash j,v)\right] \\
&= \frac{\alpha_i\alpha\omega(S)-\alpha_i^2\omega_i}{\alpha\omega(S)^2}\left(v(S)-v(S\backslash i)\right) - \sum_{j\in S\backslash i}\frac{\alpha_i\alpha_j\omega_j}{\alpha\omega(S)^2} Sh_i^{\alpha\omega}(S\backslash j,v) \\
&\quad + \sum_{j\in S\backslash i}\frac{\alpha_j\omega_j}{\alpha\omega(S)}\frac{\partial}{\partial \omega_i} Sh_i^{\alpha\omega}(S\backslash j,v)\ .
\end{aligned}
$$

In neither derivatives is there any way to make an induction assumption as to the sign of $\frac{\partial}{\partial\alpha_i}Sh_i^{\alpha\omega}(S\backslash j,v)$ and $\frac{\partial}{\partial\omega_i}Sh_i^{\alpha\omega}(S\backslash j,v)$ in order to deduce univocally the sign of $\frac{\partial}{\partial\alpha_i}Sh_i^{\alpha\omega}(S,v)$ and $\frac{\partial}{\partial\omega_i}Sh_i^{\alpha\omega}(S,v)$, even for the case of symmetric games. Therefore (E.1) and (E.2) hold. Moreover, applying induction, if it is assumed that

$$
\frac{1}{\omega_i}\frac{\partial}{\partial \alpha_i} Sh_i^{\alpha\omega}(S\backslash j,v) = \frac{1}{\alpha_i}\frac{\partial}{\partial \omega_i} Sh_i^{\alpha\omega}(S\backslash j,v)\ ,
$$

holds for all $j \in S\backslash i$, we conclude that (E.3) holds. ■

This lack of determination of the sign in the payoff variation for the weighted Shapley values is in line with the earlier results of Radzik (2012) who shows that there is a problem with the interpretation of the weight system in the context of the weighted Shapley values (see Remark 4.7). It was once usual for weights to be interpreted as a measure of the "importance" or "bargaining strength" of players in the game. However, it turns out that there are monotonic games for which the behavior of the weighted Shapley value goes in opposite direction to some players' weights (Examples 4.2 and 4.3 there), i.e. greater weights correspond to lower payoffs. This point is also discussed in an early paper by Owen (1968).

For monotonic games, the differences between the solidarity and the Shapley values can be summed up as follows:

- *Weighted solidarity values.* For every monotonic game, when α_i *rises* (the probability $\frac{\alpha_i}{\alpha(S)}$ of being the proposer increases), then $Sl_i^{\alpha\omega}(S,v)$ also *rises*. Hence, player i increases its bargaining power. If the game v is also symmetric, when ω_i *rises* (the probability ρ^{ω_i} of continuing in the game decreases), then $Sl_i^{\alpha\omega}$ also *decreases*. Hence, player i decreases its bargaining power. Thus, parameters α and ω affect the payoffs of the players in *opposite* ways.

- *Weighted Shapley values.* The parameters α_i and ω_i affect the value $Sh_i^{\alpha\omega}(S,v)$ in the *same* direction because (E.3) means that the signs of $\frac{\partial}{\partial\alpha_i}Sh_i^{\alpha\omega}$ and $\frac{\partial}{\partial\omega_i}Sh_i^{\alpha\omega}$ are both positive or negative. However, depending on the game, the payoffs of each player can increase or decrease due to changes in their weights. Hence, the interpretation of weights as a measure of players' bargaining power is less clear.

4.1 Pure bargaining problems

Pure bargaining games are usually described by a pair $(d, V(N))$, where $V(N)$ is the utility feasible set attainable by unanimous agreements of all members of N, and $d \in \mathbb{R}_+^N$ is the utility feasible payoffs vector obtained in case of disagreement. The fact that no coalition other than the grand coalition can make agreements is reflected by $0 \in \partial V(S)$, for all $S \neq N$. Therefore, the unanimity game of the grand coalition, (N, u_N), can also be considered as a particular transferable utility case of pure bargaining games, where the disagreement point is $0 \equiv (0, - - -, 0)$; that is $(N, u_N) \equiv (0, u_N)$.

It is simple to check that for the particular case of the *unanimity game* of the grand coalition, (N, u_N), it holds that

$$Sl_i^{\alpha\omega}(N,u_N) = Sh_i^{\alpha\omega}(N,u_N) = \frac{\alpha_i}{\alpha(N)} \quad (i \in N) \ .$$

This is because every subgame (S, u_N), $S \neq N$, is a zero game,[6] so all players obtain $Sl_i^{\alpha\omega}(S,u_N) = Sh_i^{\alpha\omega}(S,u_N) = 0$, for all $i \in S \subsetneq N$.

In the bargaining of the grand coalition, each player has the chance $\alpha_i/\alpha(N)$ of being the proposer, and if even one is defeated after a rejection all players obtain zero. Thus, after the rejection of a proposal made by player i the probability of bargaining continuing is $\prod_{j\in N} \rho^{\omega_j}$ for the weighted solidarity value, and ρ^{ω_i} for the weighted Shapley value. As far as $\rho \to 1$, $\prod_{i\in N} \rho^{\omega_i}$ and ρ^{ω_i} both converge to one, regardless of the weight vector ω. For this reason, ω does not influence the final payoffs in (N, u_N) and the payoffs only depend on the relative weights of α.

Therefore, in the unanimity game (N, u_N) the two solutions behave is the same way: there is a positive relationship between α and the bargaining power and no relationship at all with parameter ω. This is set out in the following theorem, the proof of which is straightforward.

Theorem 6 *Given the unanimity game (N, u_N), it holds for all $i \in N$,*

$$\begin{aligned}
&\textit{(F.1)}\ \frac{\partial}{\partial\alpha_i}Sl_i^{\alpha\omega}(N,u_N) &=& \frac{\partial}{\partial\alpha_i}Sh_i^{\alpha\omega}(N,u_N) = \frac{\alpha(N\backslash i)}{\alpha(N)} > 0 \ , \\
&\textit{(F.2)}\ \frac{\partial}{\partial\omega_i}Sl_i^{\alpha\omega}(N,u_N) &=& \frac{\partial}{\partial\omega_i}Sh_i^{\alpha\omega}(N,u_N) = 0 \ .
\end{aligned}$$

[6] A zero game (N, z) is defined by $z(S) = 0$ for all $S \subseteq N$.

Notice that the weighted split solution E^{α} is just the weighted Nash bargaining solution N^{α} (Nash, 1950), defined by

$$N^{\alpha}(d, V(N)) = \arg \max_{x \in V(N)} \prod_{i \in N} (x_i - d_i)^{\alpha_i} \ .$$

that is, it holds that

$$E_i^{\alpha}(N, u_N) = N_i^{\alpha}(0, u_N) = \frac{\alpha_i}{\alpha(N)} \ (i \in N) \ .$$

So, for the unanimity game (N, u_N) the coincidence $Sl^{\alpha\omega}(N, u_N) = Sh^{\alpha\omega}(N, u_N) = E^{\alpha}(N, u_N) = N^{\alpha}(0, u_N)$ is obtained.

5 Axiomatic characterization

This section provides an axiomatic characterization of the weighted solidarity value and the weighted Shapley value. A standard approach when asymmetric values are considered is to drop the symmetry axiom from the set which characterizes the value and then determine the whole family of values that appear. Our approach is different because we specify by means of parameters α and ω what such asymmetries consist of. For that reason, we do not drop the symmetry axiom but we "weight" the axiom used in the characterization of the symmetric value. That is, these weights appear explicitly in the formulation of the axioms. Therefore, we do not try to determine what the whole family of Shapley and solidarity values that could appear without symmetry are. Instead, we seek to identify which values appear when the differences between players are given only by the probabilities specified by parameters α and ω.

5.1 Weighted Solidarity values

We look first at the characterization of the solidarity value given by Nowak and Radzik (1994), where a variation of the null player axiom is introduced as follows: Player $i \in N$ is an *A-null player* in (N, v) if $\Delta^{av}(v, S) = 0$ for all coalitions $S \subseteq N$ containing i. The solidarity value satisfies the following axiom in G^N:

A-Null player axiom: For all (N, v) and all $i \in N$, if i is an A-null player, then $\gamma_i(N, v) = 0$.

Consider the following properties of a value γ in G^N:

Efficiency: For all (N, v), $\sum_{i \in N} \gamma_i(N, v) = v(N)$.

Additivity: For all (N, v) and (N, v'), $\gamma(N, v + v') = \gamma(N, v) + \gamma(N, v')$.

Symmetry: For all (N, v) and all $\{i, j\} \subseteq N$, if i and j are symmetric players in (N, v), then $\gamma_i(N, v) = \gamma_j(N, v)$.

Null player axiom: For all (N, v) and all $i \in N$, if i is a null player in (N, v), then $\gamma_i(N, v) = 0$.

The following theorem is due to Nowak and Radzik (1994).

Theorem 7 *[Nowak and Radzik, 1994] A value γ on G^N satisfies* efficiency, additivity, symmetry *and the* A-null player axiom *if, and only if, γ is the solidarity value.*

Compare this theorem with the standard characterization of the Shapley value:

Theorem 8 *[Shapley, 1953b] A value γ on G^N satisfies* efficiency, additivity, symmetry *and the* null player axiom *if, and only if, γ is the Shapley value.*

It is clear that the two values differ only in their treatment of null players. The null player axiom says that if all the marginal contributions of a player in a game are zero (hence it is not a productive player) then it should obtain zero. The interpretation of the A-null player is less evident. Notice that $\Delta^{av}(v,S) = 0$ means that the expected productivity of the players in coalition S is zero, as

$$\Delta^{av}(v,S) = \frac{1}{s}\sum_{i\in S}(v(S) - v(S\backslash i))$$

is the expected variation in the worth of coalition S when every player in S has the same chance $1/s$ of leaving the game. The A-null player axiom says that when the average productivity of all coalitions to which a player belongs is zero, then it *must receive zero.* But notice that the spirit behind the solidarity value is based on a type of cohesion principle which is difficult to express in terms of individual productivity alone. We therefore present an alternative way of formulating the idea that all players are "in the same boat".

Suppose that each player has the same chance of leaving the game, the expression below

$$E\left[\Delta\gamma_i(N,v)\right] = \frac{1}{n}\sum_{k\in N}(\gamma_i(N,v) - \gamma_i(N\diagdown k,v))$$

can be interpreted as the *expected variation in the payoffs of player i when each player in coalition N has the same chance of leaving the game.* Note that when player i leaves the game it obtains zero, so we can define $\gamma_i(N\diagdown i,v) := 0$. Equivalently, $E\left[\Delta\gamma_i(N,v)\right]$ measures the expected marginal contribution of the society (the rest of the players, each with the same probability $1/n$) to the payoffs of player i. We express a cohesion-type rule between players by the equality in the expected marginal contribution of the society to each player in terms of payoffs:

Definition 1 Equal average gains*. For all (N,v) and all $\{i,j\} \subseteq N$,*

$$E\left[\Delta\gamma_i(N,v)\right] = E\left[\Delta\gamma_j(N,v)\right].$$

Calvo and Gutiérrez-López (2013) offers a new characterization of the solidarity value with the help of this axiom.

Theorem 9 *A value γ on G satisfies* efficiency *and equal averaged gains if, and only if, γ is the solidarity value.*

We now consider the asymmetric case. In our setting the asymmetries are specified only by parameters $\alpha, \omega \in \mathbb{R}^N_{++}$. Accordingly, we define

$$E_{\alpha\omega}\left[\Delta\gamma_i(N,v)\right] = \sum_{k\in N}\frac{\omega_k}{\omega(N)}\frac{1}{\alpha_i}(\gamma_i(N,v) - \gamma_i(N\diagdown k,v)) \ .$$

$E_{\alpha\omega}\left[\Delta\gamma_i(N,v)\right]$ is the *expected variation in the normalized payoffs of player i when each of the players j in coalition N has the probability* $\frac{\omega_j}{\omega(N)}$ *of leaving the game.* The weighted version of the equal average gains axiom is:

Definition 2 Equal weighted average gains*. For all* (N,v) *and all* $\{i,j\}\subseteq N$*,*

$$E_{\alpha\omega}\left[\Delta\gamma_i(N,v)\right]=E_{\alpha\omega}\left[\Delta\gamma_j(N,v)\right]\ .$$

The property says that the expected marginal contributions of the society to the normalized (per-unit-weight α_i) payoffs of each player i are equal. We now offer a new characterization of the weighted solidarity value with the help of this axiom.

Theorem 10 *Given vectors* $\alpha,\omega\in\mathbb{R}^N_{++}$*. A value* γ *on* G *satisfies* efficiency *and* equal *weighted averaged gains if, and only if,* γ *is equal to* $Sl^{\alpha\omega}$*.*

Proof. *Existence.* $Sl^{\alpha\omega}$ satisfies *efficiency* by construction. Moreover, by

$$Sl_i^{\alpha\omega}(S,v)=\frac{\alpha_i}{\alpha(S)}\Delta_\omega^{av}(v,S)+\sum_{k\in S\setminus i}\frac{\omega_k}{\omega(S)}Sl_i^{\alpha\omega}(S\backslash k,v),\quad (i\in S\subseteq N)\ ,$$

starting with

$$Sl_i^{\alpha\omega}(\{i\},v)=v(i),\ \text{ for all } i\in N.$$

Therefore, for all $\{i,j\}\subseteq N$:

$$\begin{aligned}\frac{1}{\alpha_i}\left[Sl_i^{\alpha\omega}(N,v)-\sum_{k\in N\setminus i}\frac{\omega_k}{\omega(N)}Sl_i^{\alpha\omega}(N\backslash k,v)\right] &= \frac{1}{\alpha_j}\left[Sl_j^{\alpha\omega}(N,v)-\sum_{k\in N\setminus j}\frac{\omega_k}{\omega(N)}Sl_j^{\alpha\omega}(N\backslash k,v)\right]\\ &= \frac{1}{\alpha(N)}\Delta_\omega^{av}(v,N)\ ,\end{aligned}$$

and this can be written as

$$\sum_{k\in N}\frac{\omega_k}{\omega(N)}\frac{1}{\alpha_i}\left(Sl_i^{\alpha\omega}(N,v)-Sl_i^{\alpha\omega}(N\backslash k,v)\right)=\sum_{k\in N}\frac{\omega_k}{\omega(N)}\frac{1}{\alpha_j}\left(Sl_j^{\alpha\omega}(N,v)-Sl_j^{\alpha\omega}(N\backslash k,v)\right)\ ,$$

where $Sl_i^{\alpha\omega}(N\backslash i,v):=0$. Thus, $Sl^{\alpha\omega}$ satisfies *equal weighted averaged gains.*

Uniqueness. Let γ be a value satisfying the above axioms and let $(N,v)\in G^N$. We prove $\gamma=Sl^{\alpha\omega}$ by induction over the number of players n. If $n=1$, by *efficiency*, $\gamma(\{i\},v)=Sl_i^{\alpha\omega}(\{i\},v)=v(i)$ and hence the result holds. Assume that it is true for less than n players. We now prove it for n players.

By *equal weighted averaged gains*, for all $\{i,j\}\subseteq N$:

$$\sum_{k\in N}\frac{\omega_k}{\omega(N)}\frac{1}{\alpha_i}\left(\gamma_i(N,v)-\gamma_i(N\backslash k,v)\right)=\sum_{k\in N}\frac{\omega_k}{\omega(N)}\frac{1}{\alpha_j}\left(\gamma_j(N,v)-\gamma_j(N\backslash k,v)\right)\ . \tag{7}$$

By the induction hypothesis, $\gamma_i(N\setminus k,v)=Sl_i^{\alpha\omega}(N\setminus k,v)$, for all $\{i,k\}\subseteq N$. Therefore, following (7):

$$\frac{1}{\alpha_i}\gamma_i(N,v)-\frac{1}{\alpha_j}\gamma_j(N,v)=\frac{1}{\alpha_i}\sum_{k\in N}\frac{\omega_k}{\omega(N)}\gamma_i(N\backslash k,v)-\frac{1}{\alpha_j}\sum_{k\in N}\frac{\omega_k}{\omega(N)}\gamma_j(N\backslash k,v)\ .$$

This expression yields $(n-1)$ linearly independent equations which, jointly with the *efficiency*,

$$\sum_{i\in N}\gamma_i(N,v)=v(N)\ ,$$

form an $n \times n$ linear equations system. The matrix of this system is:

$$A_n = \begin{bmatrix} \frac{1}{\alpha_1} & -\frac{1}{\alpha_2} & 0 & 0 & ... & 0 \\ 0 & \frac{1}{\alpha_2} & -\frac{1}{\alpha_3} & 0 & ... & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & \ddots & 0 \\ 0 & ... & ... & 0 & \frac{1}{\alpha_{n-1}} & -\frac{1}{\alpha_n} \\ 1 & 1 & ... & 1 & 1 & 1 \end{bmatrix}$$

We now prove that $|A_n| = \frac{\alpha(N)}{\alpha_1\alpha_2...\alpha_n}$. We proceed by induction. For $n = 2$, $|A_2| = \frac{1}{\alpha_1} + \frac{1}{\alpha_2} = \frac{\alpha_2+\alpha_1}{\alpha_1\alpha_2}$. Assume that it is true for less than n. We now prove it for n. We develop $|A_n|$ with the elements of the last column:

$$|A_n| = \frac{1}{\alpha_1...\alpha_{n-1}} + \frac{1}{\alpha_n}|A_{n-1}| = \frac{1}{\alpha_1...\alpha_{n-1}} + \frac{1}{\alpha_n}\frac{\alpha(N\backslash n)}{\alpha_1\alpha_2...\alpha_{n-1}} = \frac{\alpha(N)}{\alpha_1\alpha_2...\alpha_n} \ .$$

Therefore, $|A_n| \neq 0$, which implies that the system has only one solution. Thus, we conclude that $\gamma = Sl^{\alpha\omega}$. ■

5.2 Weighted Shapley values

Myerson (1980) offers a characterization of the Shapley value by means of the *balanced contributions axiom*. This property expresses the bilateral principle that the contribution that each player makes to another in the game must be the same. Formally:

Definition 3 Balanced contributions. *For all (N, v) and all $\{i, j\} \subseteq N$,*

$$\gamma_i(N, v) - \gamma_i(N \diagdown j, v) = \gamma_j(N, v) - \gamma_j(N \diagdown i, v) \ .$$

That is, for any two players the amount that each player stands to gain or lose by the other player's withdrawal from the game must be equal. Thus, in bargaining over the surplus the bargaining powers of each two players $\{i, j\}$ are in balance with each other, because the loss in the payoff of player j that player i can inflict by withdrawing from the game is the same as that which j can inflict on i.

We have:

Theorem 11 *[Myerson, 1980] A value γ on G satisfies* efficiency *and* balanced contributions *if, and only if, γ is the Shapley value.*

Now let $w \in \mathbb{R}^N_{++}$. In Hart and Mas-Colell (1989, Section 5) the weighted Shapley values are considered. It is proved that Sh^w has a weighted potential associated with it. A *weighted potential* is, a function $P_w : G \to \mathbb{R}$ with $P_w(\emptyset, v) = 0$, satisfying

$$\sum_{i \in N} w_i \left(P_w(N, v) - P_w(N \backslash i, v)\right) = v(N) \ , \quad (8)$$

for all games (N,v). From (8), it is simple that P_w can be computed recursively by

$$P_w(N,v) = \frac{1}{w(N)}\left[v(N) + \sum_{i\in N} w_i P_w(N\backslash i, v)\right] .$$

It was proved that (Theorem 5.2)

$$Sh^w(N,v) = w_i\left(P_w(N,v) - P_w(N\backslash i, v)\right), \quad (i \in N) .$$

Moreover, the property of the *weighted preservation of differences* is satisfied by Sh^w, that is

$$\frac{1}{w_i}\gamma_i(N,v) - \frac{1}{w_j}\gamma_j(N,v) = \frac{1}{w_i}\gamma_i(N\diagdown j, v) - \frac{1}{w_j}\gamma_j(N\diagdown i, v), \quad (i,j \in N) .$$

The above expression can be rewritten as a weighted version of the balanced contributions axiom:

$$\frac{1}{w_i}\left(\gamma_i(N,v) - \gamma_i(N\diagdown j, v)\right) = \frac{1}{w_j}\left(\gamma_j(N,v) - \gamma_j(N\diagdown i, v)\right), \ (i,j \in N) . \tag{9}$$

That is, the normalized (per-unit-weight) marginal contributions between each pair of players must be equal. With weighted balanced contributions and efficiency, a straightforward adaptation of Theorem 11 can be given for the value Sh^w.

An alternative probabilistic version of (9) is given by

$$\frac{w_j}{w(N)}\left(\gamma_i(N,v) - \gamma_i(N\diagdown j, v)\right) = \frac{w_i}{w(N)}\left(\gamma_j(N,v) - \gamma_j(N\diagdown i, v)\right) ,$$

where $\frac{w_i}{w(N)}$ is the probability of player $i \in N$ leaving the game.

However, in the setting of the value $Sh^{\alpha\omega}$, the probabilities of players leaving the game are determined only by the parameter ω. Hence, an alternative expression for (9) can be considered as follows:

Definition 4 Weighted normalized balanced contributions. *Given vectors* $\alpha, \omega \in \mathbb{R}^N_{++}$*. For all* (N,v) *and all* $\{i,j\} \subseteq N$,

$$\frac{\omega_j}{\omega(N)}\frac{1}{\alpha_i}\left(\gamma_i(N,v) - \gamma_i(N\diagdown j, v)\right) = \frac{\omega_i}{\omega(N)}\frac{1}{\alpha_j}\left(\gamma_j(N,v) - \gamma_j(N\diagdown i, v)\right) .$$

Now, $\frac{1}{\alpha_i}\left(\gamma_i(N,v) - \gamma_i(N\diagdown j, v)\right)$ is the normalized (per-unit-weight) marginal contribution of player j to the payoff of player i, and $\frac{\omega_j}{\omega(N)}$ is the probability of player j leaving the game.

With this axiom, the characterization of $Sh^{\alpha\omega}$ is as follows:

Theorem 12 *[Hart and Mas-Colell, 1989] Given vectors* $\alpha, \omega \in \mathbb{R}^N_{++}$*. A value* γ *on* G *satisfies* efficiency *and* weighted normalized balanced contributions *if, and only if,* γ *is equal to* $Sh^{\alpha\omega}$.

6 Final remarks

The multilateral bargaining considered in this paper enables the weighted versions of the Shapley and solidarity values to be compared easily. They coincide in the probability of a player being chosen as the proposer in an active coalition. The only difference lies in what happens after a proposal is rejected: with the weighted Shapley value only the proposer i has a probability of being defeated, and then only coalitions N and $N \backslash i$ have a chance to be the new active set of bargaining players; whereas for the weighted solidarity value every player can leave after rejection, and then every coalition $S \subseteq N$ has a chance to be the new active set.

From an axiomatic point of view, the consequence of this difference in the breakdown is that the weighted Shapley value satisfies the null player axiom but the weighted solidarity value does not. The source of this difference is not given by the requirement of unanimity in the agreement (equivalently, the veto right). Instead, it is given by the differences in the opportunity to put the remaining players into an ultimatum situation: If an offer is rejected the proposer leaves the game and then the respondents lose its marginal contribution. In the case of the Shapley value only the proposer leaves, whereas in the weighted solidarity value all subsets of players can leave. Therefore, the power of this ultimatum position is spread from the proposer (in the Shapley value) to all players, i.e. to the proposer and respondents (in the solidarity value). In our axiomatic characterizations we are only concerned with the marginal contributions between players, i.e. $\gamma_i(N,v) - \gamma_i(N \backslash j, v)$ for each $i,j \in N$. For the weighted Shapley value these marginal contributions are *balanced*: what one player contributes to the others is the same as what the others contribute to it. This is a way of expressing a *marginality* principle: players are rewarded according to their productivity. However, for the weighted solidarity value what one player contributes to the others is *equalized* among them all. This is a type of *egalitarian* principle: players are rewarded in such a way that they all contribute the same.[7] As we are dealing with asymmetric players here, these properties must be rewritten with the corresponding weights α and ω.

In the strategic approach both values give expectations of how much players can obtain in the game. The bargaining power of each player is conditioned by the parameters α and ω, which determine the probabilities of being the proposer and leaving the game respectively.

In the pure bargaining case only the grand coalition N matters: as soon as one player leaves the game they all obtain zero. This is the case of the unanimity game (N, u_N). Here, both values yield the same payoffs:

$$Sh_i^{\alpha\omega}(N,u_N) = Sl_i^{\alpha\omega}(N,u_N) = \frac{\alpha_i}{\alpha(N)}, \quad (i \in N) .$$

The expected payoff of each player is directly related to its probability of being selected as a proposer. A greater probability implies a greater bargaining power.

However when partial agreements are allowed, i.e. $v(S) \neq 0$ for $S \subsetneq N$, the probabilities of leaving the game also affect the player's expectations. But now the two values differ in the effect on bargaining power of a variation in parameters α and ω.

In the case of the weighted Shapley value, and for generic monotonic games, the answer to this question is a little puzzling. It depends on the characteristic function, even for the particular case of symmetric games (where players are indistinguishable by their marginal contributions). All that is known is that the variation in the expected payoffs from changes in the two parameters goes in the *same* direction: if the impact in the payoffs from an increased probability $\frac{\alpha_i}{\alpha(N)}$ of being the proposer is positive then the

[7] An extreme form of egalitarianism is given by the principle of rewarding all players equally, which is given by the equal split solution.

impact of increasing the probability $\frac{\omega_i}{\omega(N)}$ of leaving the game is also positive. Both affect the bargaining power of the player in the same way, but it is not known in advance if they are positive or negative.

The case of the weighted solidarity value is rather different. For generic monotonic games the expected payoffs increase when the probability of being the proposer increases (as for pure bargaining games). The effect of changes in the probability of leaving the game is undetermined in general, but for symmetric games we find that increasing the probability of leaving the game has the effect of decreasing the expected payoffs of the player. Hence the bargaining power changes in *opposite* directions: it increases when the probability of being the proposer increases and decreases when the probability of remaining in the bargaining decreases.

In summary, from a strategic (positive) point of view, the expected payoffs of the weighted solidarity value (and the corresponding bargaining power) which players have in multilateral bargaining seem more in line with intuition than the payoffs obtained with the weighted Shapley value. On the other hand, from an axiomatic (normative) point of view, the two values have different but clear supports. The weighted Shapley value rewards players in such a way that each player contributes to each other the same as each other contributes to it. The weighted solidarity value makes society contribute to all players equally. Obviously, context decides which value must be applied, so that players are either rewarded according to their contributions, as with the Shapley value, or in an egalitarian way, as with the solidarity value.

The undetermined monotonicity obtained for the parameters $\alpha\omega$ in the weighted Shapley value is rather unsatisfactory. Intuition suggests that every bargaining game which tries to model more or less realistic negotiations should yield a positive correlation between bargaining power and the probability of being the proposer. Thus, if the results obtained for the value $Sh^{\alpha\omega}$ are disappointing what is wrong here: the specification of the model or our intuition?

Before answering this question, we wish to remark that there are alternative ways of defining weighted versions of the Shapley value. We show one of them, which we denote by $\phi^{\alpha\omega}$. This is defined recursively by

$$\phi_i^{\alpha\omega}(S,v) = \frac{\frac{\alpha_i}{\omega_i}}{\sum_{j\in S}\frac{\alpha_j}{\omega_j}} v(S) - \frac{1}{s} v(S\backslash i) + \frac{1}{s}\sum_{j\in S\backslash i}\phi_i^{\alpha\omega}(S\backslash j, v), \quad (i \in S \subseteq N)\ , \tag{10}$$

starting with $\phi_i^{\alpha\omega}(\{i\}, v) = v(i)$.

The value $\phi^{\alpha\omega}$ is homogeneous in the parameters α and ω. Moreover, when $\frac{\alpha_i}{\omega_i} = \frac{\alpha_j}{\omega_j}$ for all $i, j \in N$, $\phi^{\alpha\omega}$ is just the Shapley value.

Now, with respect to α:

$$\frac{\partial}{\partial\alpha_i}\phi_i^{\alpha\omega}(S,v) = \frac{\frac{1}{\omega_i}\sum_{j\in S\backslash i}\frac{\alpha_j}{\omega_j}}{\left(\sum_{j\in S}\frac{\alpha_j}{\omega_j}\right)^2} v(S) + \frac{1}{s}\sum_{j\in S\backslash i}\frac{\partial}{\partial\alpha_i}\phi_i^{\alpha\omega}(S\backslash j, v)\ .$$

Assume by induction that $\frac{\partial}{\partial\alpha_i}\phi_i^{\alpha\omega}(S\backslash j, v) \geq 0$ for all $j \in S\backslash i$. Then, it holds that $\frac{\partial}{\partial\alpha_i}\phi_i^{\alpha\omega}(S,v) \geq 0$.

With respect to ω:

$$\frac{\partial}{\partial\omega_i}\phi_i^{\alpha\omega}(S,v) = -\frac{\frac{\alpha_i}{\omega_i^2}\sum_{j\in S\backslash i}\frac{\alpha_j}{\omega_j}}{\left(\sum_{j\in S}\frac{\alpha_j}{\omega_j}\right)^2} v(S) + \frac{1}{s}\sum_{j\in S\backslash i}\frac{\partial}{\partial\omega_i}\phi_i^{\alpha\omega}(S\backslash j, v)\ .$$

Assume by induction that $\frac{\partial}{\partial \omega_i}\phi_i^{\alpha\omega}(S\backslash j, v) \leq 0$ for all $j \in S\backslash i$. Then it holds that $\frac{\partial}{\partial \omega_i}\phi_i^{\alpha\omega}(S, v) \leq 0$.

This is a nice monotonic behavior of the value with respect to these parameters: the bargaining power increases when the probability of being the proposer rises, and the bargaining power decreases when the probability of leaving the game increases.

Unfortunately, the value $\phi^{\alpha\omega}$ is not a good proposal for an expected value in a bargaining negotiation, as it can yield negative payoffs for adequate configurations of parameters α and ω. Any plausible bargaining protocol that can be imagined should give every player the possibility of rejecting unsatisfactory offers, and thus guarantee a payoff of at least zero in any monotonic game.

The purpose of considering this value $\phi^{\alpha\omega}$ is to remark that a value cannot be justified only axiomatically, with properties which could sound more or less appealing: it must also be supported with a reasonable bargaining model. In addition, from the analysis of the monotonicity properties of the weighted Shapley and solidarity values provided in the present work, it should be evident that the incorporation of asymmetric players into the model is not merely an academic exercise but brings to light problems of interpretation of these values which would otherwise remain hidden in the symmetric setting. The analysis of the monotonicity behavior of a particular value can yield relevant information for deciding how far this value could be a reasonable payoff expectation for players involved in a real bargaining negotiation which takes place in a cooperative setting.

The main conclusion from the monotonicity analysis of the two values with respect to the probabilities of players in the bargaining process is that the bargaining protocol associated with the solidarity value seems more realistic than that of the protocol associated with the Shapley value.

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