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Abstract

We consider cooperative games in which the cooperation among players is restricted by a *set system*, which outlines the set of feasible coalitions that actually can be formed by players in the game. In our setting, the structure of this set system is completely free, and the only restriction is that the empty set belongs to it. An extension of the Shapley value is provided as the sum of the dividends that players obtain in the game. In this general setting, we offer two axiomatic characterizations for the value: one by means of component efficiency and fairness, and the other one with efficiency and balanced contributions.

JEL classification: C71

Keywords: TU games; Restricted cooperation; Shapley value.

1 Introduction

A cooperative game with side payments is a pair (N, v), where N is a finite set of players and $v : 2^N \to \mathbb{R}$ is a characteristic function. A *restriction* in cooperation is given by

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a *set system* $\mathcal{F}$, where $\mathcal{F} \subseteq 2^N$, is the set of feasible coalitions that can be formed. A game with a restriction is is a triple $(N, \mathcal{F}, v)$. The literature considers a range of different restrictions. Some examples include: *coalition structures* (Aumann and Dreze, 1974), *communication graphs* (Myerson, 1977), *conference structures*, (Myerson, 1980), *precedence constraints* (Faigle and Kern, 1992), *conjunctive permission structures* (Gilles *et al.* 1992, and van den Brink and Gilles, 1996), *disjunctive permission structures* (Gilles and Owen, 1994, and van den Brink, 1997) *convex geometries* (Bilbao, 1998, and Bilbao and Edelman, 2000), *union stable cooperation structures* (Algaba *et al.* 2000, and 2001), *matroids* (Bilbao *et al.* 2001), *antimatroids* (Algaba *et al.* 2003), *augmenting systems* (Bilbao and Ordóñez, 2008), *union closed systems* (van den Brink *et al.* 2011), *accessible union stable systems* (Algaba *et al.* 2013), *regular set systems* (Lange and Grabisch, 2009) and *spectrum structures* (Álvarez-Mozos *et al.* 2013).

The above list is not exhaustive and continues to grow. Differences in their corresponding set system $\mathcal{F}$ determine the differences between them; and some are generalizations of previous ones. A recursive question that immediately emerges as soon as a new structure appears, is how to extend the *Shapley value* (Sh) (see Shapley, 1953) on it. We review briefly two of the approaches followed.

The first approach extends the characteristic function v to all unfeasible sets with the help of a new function $\bar{v}_{\mathcal{F}}$ computing $\bar{v}_{\mathcal{F}}(S)$ as the sum of the coalition's worth of their maximal feasible subcoalitions (called the components of the coalition). In this way, $\bar{v}_{\mathcal{F}}$ is extended on all 2^N and then the Shapley value extension ϵ^{Sh} of the game v is defined by $\epsilon^{Sh}(N, \mathcal{F}, v) := Sh(N, \bar{v}_{\mathcal{F}})$. This approach is successful for coalition structures, communication graphs, conference structures, partition systems, conjunctive and disjunctive permission structures, antimatroids, union closed systems and accessible union stable systems. For example, let $\mathcal{F} = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}\}$ be the set system. Then $\bar{v}_{\mathcal{F}}$ compute the worth of the grand coalition $\{1, 2, 3\}$ as $\bar{v}_{\mathcal{F}}(\{1, 2, 3\}) = v(\{1, 2\}) + v(3)$, the worth of $\{1, 3\}$ as $\bar{v}_{\mathcal{F}}(\{1, 3\}) = v(1) + v(3)$, and so one. In all structures mentioned previously $\bar{v}_{\mathcal{F}}$ is well defined because the components of every non feasible coalition form a partition of some subset of the coalition.

The main problem that this construction has is that it is not clear how to define $\bar{v}_{\mathcal{F}}$ for more arbitrary structures. For example, consider $\mathcal{F} = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{2, 3\}\}$, where players *1* and *3* are incompatible, so they cannot belong to a feasible coalition at the same time. Here, the set of maximal feasible subcoalitions of $\{1, 2, 3\}$ is $\{\{1, 2\}, \{2, 3\}\}$ which does not form a partition of any subset of $\{1, 2, 3\}$. Now $\bar{v}_{\mathcal{F}}$ as the sum of the maximal components of a coalition becomes a meaningless operation.

The second approach comes from the computation of the Shapley value as the player's expected marginal contributions to the coalitions formed by a sequential process: given any order of the players in N, every player joins the coalition one by one, receiving her marginal contribution to her predecessors, where all orders are equally likely. Denoting by $\Omega(N)$ the set of all orders, its cardinal is $n!$, hence every order has probability $1/n!$ of taking place. Now, given a set system $\mathcal{F}$, if a particular coalition of predecessors is not feasible in $\mathcal{F}$, the associated order will not be feasible either. In this way, we restrict to $\Omega(\mathcal{F}, N)$ the computation of the marginal contributions, each order with probability $1/|\Omega(\mathcal{F}, N)|$. This value extension $\xi(N, \mathcal{F}, v)$ applies to games with precedence constraints, convex geometries, matroids, augmenting systems, spectrum structures and regular systems.

This approach has two clear drawbacks. Firstly, by restricting the set of admissible orders compatible with the set system, we leave out of consideration many interesting classes of games for which $\Omega(\mathcal{F}, N)$ is empty as, for example, is the case of pure bargaining problems, $\mathcal{F} = \{\emptyset, N, \{i\} : i \in N\}$, and team games, $\mathcal{F} = \{\emptyset, S : |S| = k\}$ where $k < n$. Second, as we will see with the help of several examples, the allocation payoffs obtained in this way can be rather counterintuitive.

Our goal is to extend the value to any arbitrary set system $\mathcal{F}$. For that purpose, we will follow a dividends approach: given any game (N, v) and any set system $\mathcal{F}$, the worth of the feasible coalitions $(v(S))_{S\in\mathcal{F}}$ can be associated one-to-one with their associated dividends $(\Delta(S, \mathcal{F}, v))_{S\in\mathcal{F}}$, where the worth of every feasible coalition is the sum of the dividends of all of their feasible subcoalitions, i.e. $v(S) = \sum_{T\in\mathcal{F}:T\subseteq S} \Delta(T, \mathcal{F}, v)$. The number $\Delta(S, \mathcal{F}, v)$ measures the additional gains that obtain members of S with their cooperation with respect to the sum of all the gains that they have obtained previously. The Shapley value is a fair rule, which equally shares these gains among the members of the coalition. Our extended Shapley value yields the sum of all the gains that the player obtains for participation in all the *feasible* coalitions for which she belongs:

$$\gamma_i(N, \mathcal{F}, v) = \sum_{S\in\mathcal{F}:\ i\in S} \frac{1}{s}\Delta(S, \mathcal{F}, v) \ .$$

In addition, it is zero when she does not belong to any feasible coalition.

In the rest of this paper, we offer an axiomatic support for this value and a short comparison with the main existing value alternatives.

Following this introduction, in Section 2 we introduce some preliminary concepts and the definition of the value by means of dividends. In Section 3, we provide two axiomatic characterizations of the value. One is by means of *component efficiency* and *fairness*, and the other one with *efficiency* and *balanced contributions.* Section 4 is devoted to some additional remarks.

First, we show that $\gamma(N,\mathcal{F},v) = Sh(N,\bar{v}_{\mathcal{F}})$ for the cases of coalition structures, communication graphs, conference structures, partition systems, augmenting systems and accessible union stable systems. Hence, in all of them our value γ coincides with the previous approach given by ϵ^{Sh}, that is $\gamma(N,\mathcal{F},v) = \epsilon^{Sh}(N,\mathcal{F},v) = Sh(N,\bar{v}_{\mathcal{F}})$.

Second, we show in some examples that γ yield more natural payoffs than the extension $\xi(N,\mathcal{F},v)$, defined as the expected marginal contributions of the players into the set of orders $\Omega(\mathcal{F},N)$ compatible with $\mathcal{F}$.

Third, the property of *stability* is considered. This property says that in superadditive games it is always beneficial for the members of a coalition to be feasible. We show that for arbitrary set systems the answer is not determined in advance, even for superadditive games. Nevertheless, we find a situation where belonging to $\mathcal{F}$ is neutral: if $\Delta(S,\mathcal{F},v) = 0$ for $S \in \mathcal{F}$ then $\gamma(N,\mathcal{F},v) = \gamma(N,\mathcal{F}\backslash S,v)$. That is, if the feasibility of a coalition S does not sum additional gains with respect to gains obtained previously without it, then the deletion of this coalition from $\mathcal{F}$ should not affect the final payoffs. We call it *the inessential coalition out property.* We also show that γ satisfies the property of *independence of irrelevant coalitions*, which states that the payoffs only depend on the worth of feasible coalitions.

In the last remark, we show that if we wish to extend a solution into the setting of restricted cooperation preserving both aforementioned properties, we are restricted to following the dividends approach. Specifically, for every solution φ defined on 2^N, an extended solution ϵ^{φ} defined on $\mathcal{F}$, satisfies the properties of *inessential coalition out* and *independence of irrelevant coalitions* if, and only if, $\epsilon^{\varphi}(N,\mathcal{F},v) = \varphi(N,v_{\mathcal{F}})$ where

$$v_{\mathcal{F}}(S) = \sum_{T\in\mathcal{F}:T\subseteq S} \Delta(T,\mathcal{F},v), \quad S \subseteq N\ .$$

As examples, we extend two well-known families of values: the selectope, and the efficient, linear and symmetric values.

2 The extended Shapley value

Let $U = \{1, 2, ...\}$ be the (infinite) set of potential players. A *cooperative game* with transferable utility (TU-game) is a pair (N, v) where $N \subseteq U$ is a nonempty and finite set and $v : 2^N \longrightarrow \mathbb{R}$ is a *characteristic function* satisfying $v(\emptyset) = 0$, where 2^N is the set of all subsets of N. Every nonempty subset S of N is called *a coalition* and the real number $v(S)$, the *worth* of S. We denote by $s = |S|$. A game (N, v) is *superadditive* if $v(S \cup T) \geq v(S) + v(T)$ whenever $S \cap T = \emptyset$. A game (N, v) is *monotonic* if $v(S) \geq v(T)$ whenever $T \subseteq S$[1]. Two players $\{i, j\} \subseteq N$ are symmetric in (N, v) if $v(S \cup i) = v(S \cup j)$, for each $S \subseteq N \backslash \{i, j\}$. A player $i \in N$ is a *null player* in game (N, v) if $v(S \cup i) = v(S)$ for all $S \subseteq N \backslash i$. We denote by G the set of all games, and by G^N the set of all games with player set N. Given a game (N, v) and $S \subseteq N$, we denote by (S, v) the natural restriction of (N, v) to S only (i.e. to 2^S).

A *value* is a function ψ which assigns a real number $\psi_i(N, v)$ to every game (N, v) and every player $i \in N$. Since partial cooperation within subcoalitions $S \subseteq N$ is also possible, we can consider a value as a *payoff configuration* $\psi(v) = (\psi(S, v))_{S \subseteq N}$.

Some properties that a value ψ could satisfy on G are

Efficiency: $\sum_{i \in N} \psi_i(N, v) = v(N)$ for all $(N, v) \in G^N$.

Symmetry: $\psi_i(N, v) = \psi_j(N, v)$ whenever i and j are symmetric players in $(N, v) \in G^N$.

Linearity: $\psi(N, \alpha v + \beta w) = \alpha \psi(N, v) + \beta \psi(N, w)$ for all $(N, v), (N, w) \in G^N$ and $\alpha, \beta \in \mathbb{R}$.

Null Player: $\psi_i(N, v) = 0$ whenever i is a null player in $(N, v) \in G^N$.

A unique value satisfies these four properties. This is the Shapley value (Shapley, 1953), defined by

$$Sh_i(N, v) = \sum_{S \subseteq N \backslash i} \frac{s!\,(n - s - 1)!}{s!} \left[v(S \cup i) - v(S) \right], \quad i \in N, \ (N, v) \in G^N .$$

[1] We denote by $\subset$ the strict inclusion. We also note by s the cardinality of S when no confusion can arise.

The random order approach bases an alternative definition. Let $\Omega(N)$ be the set of all orders on N. Each $\omega \in \Omega(N)$ is a bijection from N to N, and $\omega(i) < \omega(j)$ means that player i comes before j in the order ω. We denote by P_ω^i the set of predecessors of i in ω, that is $P_\omega^i := \{j \in N : \omega(j) < \omega(i)\}$. We define the *marginal contribution* that every player $i \in N$ receives in any order $\omega \in \Omega(N)$ as

$$m_i^\omega(N, v) := v\left(P_\omega^i \cup i\right) - v\left(P_\omega^i\right).$$

We have then:

$$Sh_i(N, v) = \frac{1}{n!} \sum_{\omega \in \Omega(N)} m_i^\omega(N, v), \quad i \in N. \tag{1}$$

The payoff $Sh_i(N, v)$ is the expected marginal contribution of player i with respect to a uniform distribution over all orders on N.

Every game (N, v) has associated a unique *dividends collection* $\Delta(v) = (\Delta(S, v))_{S \subseteq N}$ in the following way: starting with $\Delta(\emptyset, v) = 0$, and defined recursively by

$$\Delta(S, v) = v(S) - \sum_{T \subset S} \Delta(T, v), \quad S \subseteq N.$$

The number $\Delta(S, v)$ is interpreted as the additional dividend (gains) the coalition S obtains, *if it is formed*, given that all proper subcoalitions of S have been already formed. These $\Delta(S, v)$ are called the Harsanyi dividends (Harsanyi, 1963), and each $\Delta(S, v)$ has to be shared among members of the coalition S. Note that the dividend is a real number that can be positive or negative, even if the game is superadditive. We say a game (N, v) is *totally positive* if $\Delta(S, v) \geq 0$ for all $S \subseteq N$. A sufficient condition to be totally positive is that (N, v) is a convex game.

We can define also the Shapley value in terms of dividends by

$$Sh_i(N, v) = \sum_{S \subseteq N:\ i \in T} \frac{1}{t} \Delta(S, v), \quad i \in N. \tag{2}$$

A *restriction* in the cooperation among the players in N is given by a *set system* $\mathcal{F}$, where $\mathcal{F} \subseteq 2^N$ is the set of *feasible* coalitions that can be formed. Hereafter we will assume that $\emptyset \in \mathcal{F}$. Given $S \subseteq N$ and $\mathcal{F} \subseteq 2^N$, we denote by $\mathcal{F}(S)$ the natural restriction of $\mathcal{F}$ into S, that is $\mathcal{F}(S) := 2^S \cap \mathcal{F}$. A game (N, v) with a restriction $\mathcal{F}$ is denoted by $(N, \mathcal{F}, v)$, where $\mathcal{F} \subseteq 2^N$ and $(N, v) \in G^N$.The set of all games with a restriction and player set N

is denoted by RG^N, and the set of all games with a restriction is denoted by RG. In what follows, we simplify the notation of $(S, \mathcal{F}(S), v_{|S})$ writing $(S, \mathcal{F}, v)$.

Given a game $(N, \mathcal{F}, v)$ we associate a feasible dividends collection $\Delta(\mathcal{F}, v) = (\Delta(S, \mathcal{F}, v))_{S \in \mathcal{F}}$, obtained recursively by

$$\Delta(S, \mathcal{F}, v) = v(S) - \sum_{T \in \mathcal{F}(S) \setminus S} \Delta(T, \mathcal{F}, v), \quad S \in \mathcal{F} \ , \tag{3}$$

starting with $\Delta(\emptyset, \mathcal{F}, v) = 0$.

A *value* on RG is a function ψ which assigns a payoff vector $\psi(N, \mathcal{F}, v) \in \mathbb{R}^N$ to every game $(N, \mathcal{F}, v)$. Since partial cooperation within subcoalitions $S \subseteq N$ is also possible, we can consider a value as a *payoff configuration* $\psi(\mathcal{F}, v) = (\psi(S, \mathcal{F}, v))_{S \subseteq N}$. The vector $\psi(S, \mathcal{F}, v) \in \mathbb{R}^S$ specifies how players in S share the gains of cooperation at their disposition *given by the restriction* $\mathcal{F}$.

Now, in order to extend the Shapley value into the RG setting we will follow the dividends approach. Firstly, we define the set of all feasible coalitions to which a player belongs to by:

$$\mathcal{F}(i, N) = \{S \in \mathcal{F} : i \in S\}, \quad i \in N \ .$$

Definition 1 *The extended Shapley value γ on RG is defined by*

$$\gamma_i(N, \mathcal{F}, v) = \begin{cases} 0 & \text{if } \mathcal{F}(i, N) = \emptyset \ , \\ \sum_{S \in \mathcal{F}(i,N)} \frac{1}{s} \Delta(S, \mathcal{F}, v) & \text{otherwise} \ , \end{cases}$$

for all $i \in N$, and $(N, \mathcal{F}, v) \in RG^N$.

As the dividends collection $\Delta(\mathcal{F}, v)$ is uniquely determined, γ is well defined.

Remark: In Hart and Mas-Colell (1989) it was shown that a potential function $P : G \to \mathbb{R}$ exists, with $P(\emptyset, v) = 0$, such that its gradient yields the Shapley value:

$$P(N, v) - P(N \setminus i, v) = Sh_i(N, v), \quad i \in N, \ (N, v) \in G^N \ .$$

Given our dividends approach, it is immediate to define a potential function associated to γ on RG. The proof of the next Proposition is left to the reader.

Proposition 1 *Let P be a function $P : RG \to \mathbb{R}$ defined by $P(N, \{\emptyset\}, v) = 0$ and*

$$P(N, \mathcal{F}, v) = \sum_{S \in \mathcal{F}} \frac{1}{s} \Delta(S, \mathcal{F}, v), \quad (N, \mathcal{F}, v) \in RG^N \ .$$

Then it holds that

$$P(N, \mathcal{F}, v) - P(N \setminus i, \mathcal{F}, v) = \gamma_i(N, \mathcal{F}, v), \quad i \in N, \ (N, \mathcal{F}, v) \in RG^N \ .$$

3 Axiomatization of γ

In this section, we will offer two characterizations for the value γ. In the first one, we use the property of *fairness* introduced in Myerson (1977), which takes care explicitly of changes in the payoffs due to changes in $\mathcal{F}$. In the second one, we use the property of *balanced contributions*, introduced by Myerson (1980), closely related with the existence of a *potential* (see Hart and Mas-Colell, 1989, and Calvo and Santos, 1997).

3.1 Component efficiency and fairness

In Myerson (1977) was given an extension of the Shapley value for the setting of games restricted by communication situations, where the restriction comes specified by all connected subgraphs of a graph. He gave an axiomatic characterization of the value with the help of the properties of *component efficiency* and *fairness*. The interest of both properties is that they are defined by explicitly taking into account the restrictions induced by the graph.

Firstly, note that within G^N setting, by the definition of dividends, it holds that $v(N) = \sum_{S \subseteq N} \Delta(S, v)$ for all $(N, v) \in G^N$. Therefore, in the same way we can write efficiency as:

Efficiency: $\sum_{i \in N} \psi_i(N, v) = \sum_{S \subseteq N} \Delta(S, v)$ for any $(N, v) \in G^N$.

We follow this last definition when we have a restriction given by $\mathcal{F}$:

(E) *Efficiency:* $\sum_{i \in N} \psi_i(N, \mathcal{F}, v) = \sum_{S \in \mathcal{F}} \Delta(S, \mathcal{F}, v)$ for any $(N, \mathcal{F}, v) \in RG^N$.

This property prescribes the full distribution of the *total gains* obtained by the players, given the *cooperation restrictions specified by* $\mathcal{F}$. In particular, when $S \in \mathcal{F}$, $\sum_{i \in S} \psi_i(S, \mathcal{F}, v)$ is the feasible worth $v(S)$.

A communication graph makes a partition of N into a set of maximal connected coalitions of players who can coordinate among themselves directly or indirectly by the communication graph. We extend this idea of connected components to every restriction given by a set system $\mathcal{F}$.

Let $\mathcal{F}$ be a set system on N. We say that two players $i, j \in N$ are *connected* by $\mathcal{F}$ if $i = j$ or a sequence $\{S_1, S_2, ..., S_k\} \subseteq \mathcal{F}$ exists such that $S_i \cap S_{i+1} \neq \emptyset$ for each $i = 1, ..., k-1$, $i \in S_1$ and $j \in S_k$. Thus, the restriction $\mathcal{F}$ induces a partition of the grand coalition N into maximal connected set of players who can cooperate in a direct or indirect way by feasible coalitions. We denote by $N/\mathcal{F}$ such partition of N into maximal connected components:

$$N/\mathcal{F} = \{\{j \in N : i \text{ and } j \text{ are connected by } \mathcal{F}\} : i \in N\}.$$

For example, let $N = \{1, 2, 3, 4\}$ and $\mathcal{F} = \{\emptyset, \{1\}, \{2, 3\}, \{3, 4\}\}$, then $N/\mathcal{F} = \{\{1\}, \{2, 3, 4\}\}$. Note that if $\mathcal{F} = \{\emptyset\}$, then $N/\mathcal{F} = \{\{i\} : i \in N\}$.

In the definition of component efficiency, we assume that such maximal components form in order to share all the maximal gains available to them.

(CE) Component efficiency: $\sum_{i\in S} \psi_i(N, \mathcal{F}, v) = \sum_{T\in\mathcal{F}(S)} \Delta(T, \mathcal{F}, v)$, for any $S \in N/\mathcal{F}$, and $(N, \mathcal{F}, v) \in RG^N$.

The principle of fairness is based on the idea that any two pair of players should obtain the same benefits (or losses) from their joint cooperation, relative to what they would get without such cooperation. As we specify the restriction in the cooperation by the set of feasible coalitions at their disposition, we define fairness as follows:

(F) Fairness: $\psi_i(N, \mathcal{F}, v) - \psi_i(N, \mathcal{F}\backslash S, v) = \psi_j(N, \mathcal{F}, v) - \psi_j(N, \mathcal{F}\backslash S, v)$, for each pair $i, j \in S \in \mathcal{F}$, and $(N, \mathcal{F}, v) \in RG^N$.

We now extend the Myerson characterization.

Theorem 1 *A value ψ on RG^N satisfies* fairness *and* component efficiency *if and only if $\psi(N, \mathcal{F}, v) = \gamma(N, \mathcal{F}, v)$ for any $(N, \mathcal{F}, v) \in RG^N$.*

Proof. *Existence.* Let $(N, \mathcal{F}, v) \in RG^N$ and $i \in S \in \mathcal{F}$. By definition,

$$\gamma_i(N, \mathcal{F}, v) - \gamma_i(N, \mathcal{F}\backslash S, v) = \sum_{T\in\mathcal{F}(i,N)} \frac{1}{t}\Delta(T, \mathcal{F}, v) - \sum_{T\in\mathcal{F}(i,N)\backslash S} \frac{1}{t}\Delta(T, \mathcal{F}\backslash S, v) .$$

Let $T \in \mathcal{F}$, with $T \not\supseteq S$, then it holds $\Delta(T, \mathcal{F}, v) = \Delta(T, \mathcal{F}\backslash S, v)$. Thus,

$$\gamma_i(N, \mathcal{F}, v) - \gamma_i(N, \mathcal{F}\backslash S, v) = \frac{1}{s}\Delta(S, \mathcal{F}, v) + \sum_{T\in\mathcal{F}:\ T\supset S} \frac{\Delta(T, \mathcal{F}, v) - \Delta(T, \mathcal{F}\backslash S, v)}{t} . \quad (4)$$

Since expression (4) does not depend on the player $i \in S$, we conclude that γ satisfies *fairness.*

In order to prove that γ satisfies *component efficiency*, let $S \in N/\mathcal{F}$ and $i \in S$. Since $N/\mathcal{F}$ is a partition of N, we have that

$$\gamma_i(N,\mathcal{F},v) = \sum_{T\in\mathcal{F}(i,N)} \frac{1}{t}\Delta(T,\mathcal{F},v) = \sum_{T\in\mathcal{F}(i,S)} \frac{1}{t}\Delta(T,\mathcal{F},v) \ .$$

Therefore,

$$\sum_{i\in S}\gamma_i(N,\mathcal{F},v) = \sum_{T\in\mathcal{F}(S)} \Delta(T,\mathcal{F},v) \ .$$

Thus, γ satisfies *component efficiency.*

Uniqueness. Let ψ be a value on RG^N satisfying the above axioms and let $(N,\mathcal{F},v) \in RG^N$. We prove $\psi \equiv \gamma$ by induction over $|\mathcal{F}|$. If $\mathcal{F} = \{\emptyset\}$, $|\mathcal{F}| = 1$, then $N/\mathcal{F} = \{\{i\} : i \in N\}$. Applying *component efficiency*, we have that $\psi_i(N,\mathcal{F},v) = \Delta(\emptyset,\mathcal{F},v) = 0$, for all $i \in N$.

Suppose that $\mathcal{F} = \{\emptyset, S\}$ for some $S \subseteq N$. Then, $N/\mathcal{F} = \{S, \{j\} : j \in N\backslash S\}$ and by *component efficiency*, we have that

$$\sum_{i\in S}\psi_i(N,\mathcal{F},v) = \Delta(\emptyset,\mathcal{F},v) + \Delta(S,\mathcal{F},v) = v(S) \ ,$$

and $\psi_j(N,\mathcal{F},v) = \Delta(\emptyset,\mathcal{F},v) = 0$, for all $j \in N\backslash S$. As $\mathcal{F}\backslash S = \{\emptyset\}$, it holds, by *fairness*, that $\psi_i(N,\mathcal{F},v) = \psi_j(N,\mathcal{F},v)$ for each $i,j \in S$, so $\psi_i(N,\mathcal{F},v) = v(S)/|S|$, for all $i \in S$.

Thus, ψ is uniquely determined when $|\mathcal{F}| = 1$ and 2. Let us assume that uniqueness is established for $|\mathcal{F}| \leq m$ and let $(N,\mathcal{F},v) \in RG^N$ be a game such that $|\mathcal{F}| = m + 1$. Let $S \in N/\mathcal{F}$. If $|S| = 1$, let $S = \{i\}$. We have by *component efficiency*, that $\psi_i(N,\mathcal{F},v) = \sum_{T\in\mathcal{F}(\{i\})}\Delta(T,\mathcal{F},v) = \gamma_i(N,\mathcal{F},v)$. Let us assume that $|S| \geq 2$ and let $i,j \in S$. Since i and j are connected by $\mathcal{F}$, there exists a sequence $\{S_1, S_2, ..., S_k\} \subseteq \mathcal{F}$ such that $S_r \cap S_{r+1} \neq \emptyset$ for each $r = 1,...,k-1$, $i \in S_1$ and $j \in S_k$. Applying *fairness* to S_r, for $r = 1,...,k$, we have that, for each $h,l \in S_r$,

$$\psi_h(N,\mathcal{F},v) - \psi_h(N,\mathcal{F}\backslash S_r,v) = \psi_l(N,\mathcal{F},v) - \psi_l(N,\mathcal{F}\backslash S_r,v) \ . \tag{5}$$

By induction hypothesis, we have that $\psi_h(N,\mathcal{F}\backslash S_r,v) = \gamma_h(N,\mathcal{F}\backslash S_r,v)$, for $h \in S_r$. As γ satisfies *fairness* and (5), for each $h,l \in S_r$ we have that :

$$\begin{aligned}\psi_h(N,\mathcal{F},v) - \psi_l(N,\mathcal{F},v) &= \psi_h(N,\mathcal{F}\backslash S_r,v) - \psi_l(N,\mathcal{F}\backslash S_r,v) \\ &= \gamma_h(N,\mathcal{F}\backslash S_r,v) - \gamma_l(N,\mathcal{F}\backslash S_r,v) \\ &= \gamma_h(N,\mathcal{F},v) - \gamma_l(N,\mathcal{F},v) \ .\end{aligned}$$

Thus,

$$\psi_h(N,\mathcal{F},v) - \gamma_h(N,\mathcal{F},v) = \psi_l(N,\mathcal{F},v) - \gamma_l(N,\mathcal{F},v) \ .$$

Since $S_r \cap S_{r+1} \neq \emptyset$ for every $r = 1, ..., k-1$, we conclude that

$$\psi_i(N,\mathcal{F},v) - \gamma_i(N,\mathcal{F},v) = \psi_j(N,\mathcal{F},v) - \gamma_j(N,\mathcal{F},v) \ ,$$

for each $i, j \in S$.

By *component efficiency*, we have that

$$\sum_{i\in S} \psi_i(N,\mathcal{F},v) = \sum_{i\in S} \gamma_i(N,\mathcal{F},v) \ ,$$

which implies that $\psi_i(N,\mathcal{F},v) = \gamma_i(N,\mathcal{F},v)$, for all $i \in S$. As $N/\mathcal{F}$ is a partition of N, the proof is complete. ■

3.2 Independence of the axiomatic system in Theorem 1.

1. The value ψ^1, defined as $\psi_i^1(N,\mathcal{F},v) = 0$ for all $(N,\mathcal{F},v) \in RG^N$ and $i \in N$, satisfies *fairness*, but not *component efficiency.*

2. The value ψ^2, defined as $\psi_i^2(N,\mathcal{F},v) = \left(\sum_{T\in\mathcal{F}(S)} \Delta(T,\mathcal{F},v)\right) / |S|$ for any $(N,\mathcal{F},v) \in RG^N$ and $i \in S \in N/\mathcal{F}$, satisfies *component efficiency*, but not *fairness.* Indeed, consider the following example: $N = \{1, 2, 3\}$, $\mathcal{F} = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}\}$ and $v(1) = v(3) = 1$, $v(2) = 2$, $v(\{1, 2\}) = 4$. Then, $N/\mathcal{F} = \{\{1, 2\}, \{3\}\}$, $\psi_1^2(N,\mathcal{F},v) = \psi_2^2(N,\mathcal{F},v) = \left(\sum_{T\in\mathcal{F}(\{1,2\})} \Delta(T,\mathcal{F},v)\right)/2 = 2$, and $\psi_3^2(N,\mathcal{F},v) = \left(\sum_{T\in\mathcal{F}(\{3\})} \Delta(T,\mathcal{F},v)\right) = 1$. If coalition $S = \{1, 2\}$ becomes unfeasible, $\mathcal{F}\backslash S = \{\emptyset, \{1\}, \{2\}, \{3\}\}$ and $\psi_1^2(N,\mathcal{F}\backslash S,v) = \Delta(\{1\}, \mathcal{F}\backslash S, v) = 1$, $\psi_2^2(N,\mathcal{F}\backslash S,v) = \Delta(\{2\}, \mathcal{F}\backslash S, v) = 2$. Hence, ψ^2 does not satisfy *fairness.*

3.3 Efficiency and balanced contributions

In Myerson (1980) the principle of *balanced contributions* was introduced*:*

Balanced contributions: $\psi_i(N,v) - \psi_i(N\backslash j,v) = \psi_j(N,v) - \psi_j(N\backslash i,v)$, for each pair $i, j \in N$, and $(N,v) \in G^N$.

This property states that for any two players, the amount that each player would gain or lose by the other player's withdrawal from the game should be equal. In other terms, in the bargaining over the surplus, each pair of players $\{i, j\}$ is balanced because the loss in the payoff for player j that a player i can inflict by withdrawing from the game is the same as j can inflict on i. Then we have on G:

Theorem 2 *(Myerson, 1980) A value ψ on G satisfies* efficiency *and* balanced contributions *if and only if $\psi \equiv Sh$.*

The extension of the balanced contributions definition on RG is straightforward:

(BC) *Balanced contributions:* $\psi_i(N, \mathcal{F}, v) - \psi_i(N \backslash j, \mathcal{F}, v) = \psi_j(N, \mathcal{F}, v) - \psi_j(N \backslash i, \mathcal{F}, v)$, for each pair $i, j \in N$, and $(N, \mathcal{F}, v) \in RG^N$.

We extend Theorem (2) on RG.

Theorem 3 *A value ψ on RG satisfies* efficiency *and* balanced contributions *if and only if $\psi \equiv \gamma$.*

Proof. *Existence.* Definition (1) implies the efficiency of γ. The existence of a potential implies balanced contributions:

$$\begin{aligned}
\psi_i(N, \mathcal{F}, v) - \psi_i(N \backslash j, \mathcal{F}, v) &= [P(N, \mathcal{F}, v) - P(N \backslash i, \mathcal{F}, v)] - [P(N \backslash j, \mathcal{F}, v) - P(N \backslash j \backslash i, \mathcal{F}, v)] \\
& \quad [P(N, \mathcal{F}, v) - P(N \backslash j, \mathcal{F}, v)] - [P(N \backslash i, \mathcal{F}, v) - P(N \backslash j \backslash i, \mathcal{F}, v)] \\
&= \psi_j(N, \mathcal{F}, v) - \psi_j(N \backslash i, \mathcal{F}, v) \ .
\end{aligned}$$

Uniqueness. Let ψ be a value on RG satisfying the above axioms and let $(N, \mathcal{F}, v) \in RG^N$. We prove $\psi \equiv \gamma$ by induction over $|N|$. If $|N| = 1$, $N = \{i\}$, then $\mathcal{F} = \{\emptyset\}$ or $\mathcal{F} = \{\emptyset, \{i\}\}$. By efficiency, if $\mathcal{F} = \{\emptyset, \{i\}\}$, then $\psi_i(\{i\}, \{\emptyset, \{i\}\}, v) = \Delta(\{i\}, \{\emptyset, \{i\}\}, v) + \Delta(\emptyset, \{\emptyset, \{i\}\}, v) = v(i) + 0 = v(i)$; and if $\mathcal{F} = \{\emptyset\}$, then $\psi_i(\{i\}, \{\emptyset\}, v) = \Delta(\emptyset, \{\emptyset\}, v) = 0$.

Thus, ψ is uniquely determined when $|N| = 1$. Let us assume that the uniqueness is established for $|N| \leq m$ and let $(N, \mathcal{F}, v) \in RG^N$ be a game such that $|N| = m + 1$. Applying balanced contributions to ψ and γ, and the induction hypothesis, for each $i, j \in N$ we have that

$$\begin{aligned}
\psi_i(N, \mathcal{F}, v) - \psi_j(N, \mathcal{F}, v) &= \psi_i(N \backslash j, \mathcal{F}, v) - \psi_j(N \backslash i, \mathcal{F}, v) \\
&= \gamma_i(N \backslash j, \mathcal{F}), v) - \gamma_j(N \backslash i, \mathcal{F}, v) \\
&= \gamma_i(N, \mathcal{F}, v) - \gamma_j(N, \mathcal{F}, v) \ ,
\end{aligned}$$

Thus, it holds that

$$\psi_i(N,\mathcal{F},v) - \gamma_i(N,\mathcal{F},v) = \psi_j(N,\mathcal{F},v) - \gamma_j(N,\mathcal{F},v) \ .$$

This expression, jointly with efficiency, implies that $\psi_i(N,\mathcal{F},v) = \gamma_i(N,\mathcal{F},v)$, for all $i \in N$. ■

3.4 Independence of the axiomatic system in Theorem 2.

1. The value ψ^1, defined as $\psi_i^1(N,\mathcal{F},v) = 0$ for any $(N,\mathcal{F},v) \in RG^N$ and $i \in N$, satisfies balanced contribution, but not efficiency.

2. Let ψ^3 be the value defined as $\psi_i^3(N,\mathcal{F},v) = \left(\sum_{T\in\mathcal{F}} \Delta(T,\mathcal{F},v)\right)/|N|$ for any $(N,\mathcal{F},v) \in RG^N$ and $i \in N$. ψ^3 satisfies efficiency, but does not satisfy balanced contributions. Indeed, consider the following example: $N = \{1,2,3\}$, $\mathcal{F} = \{\emptyset,\{1\},\{2\},\{3\},\{1,2\}\}$ and $v(1) = v(3) = 1$, $v(2) = 2$, $v(1,2) = 4$. Then, it holds that $\psi_1^3(N,\mathcal{F},v) = \psi_2^3(N,\mathcal{F},v) = \psi_3^3(N,\mathcal{F},v) = \left(\sum_{T\in\mathcal{F}} \Delta(T,\mathcal{F},v)\right)/3 = 5/3$. Nevertheless, $\mathcal{F}(N\backslash 2) = \{\emptyset,\{1\},\{3\}\}$ and $\mathcal{F}(N\backslash 1) = \{\emptyset,\{2\},\{3\}\}$, therefore $\psi_1^3(N\backslash 2,\mathcal{F},v) = 1$ and $\psi_2^3(N\backslash 1,\mathcal{F},v) = 3/2$. Hence, ψ^3 does not satisfy *balanced contributions.*

3.5 Comparison between the axiomatic characterizations

Note that *component efficiency* (CE) implies *efficiency* (E), but the contrary is not true. The value ψ^3, defined as $\psi_i^3(N,\mathcal{F},v) = v_{\mathcal{F}}(N)/|N|$ for any $(N,\mathcal{F},v) \in RG^N$ and $i \in N$, satisfies *efficiency*, but not *component efficiency.* Hence, the following inclusion holds: $\{\psi : \text{CE}\} \subset \{\psi : \text{E}\}$. Moreover, in the presence of *fairness* (F), both versions of efficiency are not equivalent. Consider the solution ψ^4 defined as

$$\psi_i^4(N,\mathcal{F},v) = \frac{1}{n}\left[v_{\mathcal{F}}(N) - \sum_{S\in\mathcal{F}\backslash N} v(S)\right] + \sum_{S\in\mathcal{F}(i,N)\backslash N} \frac{1}{s} v(S) \ ,$$

for any $(N,\mathcal{F},v) \in RG^N$ and $i \in N$. It is easy to see that ψ^4 satisfies *efficiency* and *fairness*, but $\psi^4 \neq \gamma$, hence it does not satisfy *component efficiency.* Therefore, the following relationships hold:

$$\{\gamma\} = \{\psi : \text{ CE+F}\} = \{\psi : \text{ E+BC}\} \subset \{\psi : \text{ E+F}\} \ .$$

As γ satisfies CE and *balanced contributions* (BC), we have as a corollary the next characterization:

$$\{\gamma\} = \{\psi : \text{ CE+BC}\} \ .$$

Nevertheless, comparing both characterizations $\{\psi : \text{ CE+F}\}$ and $\{\psi : \text{ CE+BC}\}$, the former is better than the second as F is a weaker property than BC.

4 Final remarks

4.1 Comparison with previous approaches

When $\mathcal{F} \neq 2^N$, two main approaches have been followed trying to extend formula (1). The first approach extends the characteristic function v over all unfeasible coalitions, and then obtains a new characteristic function $\bar{v}_{\mathcal{F}}$, defined now in all subsets of 2^N. The second approach computes the marginal contributions $m_i^{\omega}(N, v)$ only in the set of orders which are compatible with $\mathcal{F}$, assuming that all orders are equally likely.

4.1.1 Extensions of the characteristic function

This approach is based on the works of Myerson (1977) for graphs, and Myerson (1980) with regard to conference structures. Roughly speaking, the idea is that when $S \notin \mathcal{F}$, $\bar{v}_{\mathcal{F}}(S)$ is computed as the maximum sum of the coalition's worth of a *feasible partition* of S, that is, players within S make a feasible partition such that the sum of the total payoffs is the biggest possible. In this way, every game $(N, \mathcal{F}, v) \in RG^N$ is converted into a new game $(N, \bar{v}_{\mathcal{F}}) \in G^N$, and the value extension ϵ^{Sh} is defined by $\epsilon^{Sh}(N, \mathcal{F}, v) = Sh(N, \bar{v}_{\mathcal{F}})$.

More precisely, let $\mathcal{F}$ be a set system on N and let $S \subseteq N$. A set $T \subseteq S$ is called a *maximal component* of S if $T \in \mathcal{F}$ and there exists no $T' \in \mathcal{F}$ such that $T \subset T' \subseteq S$. The set of maximal components of S is denoted by $C_{\mathcal{F}}(S)$. Note that if $S \in \mathcal{F}$ then $C_{\mathcal{F}}(S) = \{S\}$, moreover, if $S \notin \mathcal{F}$, $C_{\mathcal{F}}(S)$ may be the empty set.

Definition 2 *Given a set system $\mathcal{F}$, the game $\bar{v}_{\mathcal{F}} : 2^N \to \mathbb{R}$ is defined by*

$$\bar{v}_{\mathcal{F}}(S) = \sum_{T \in C_{\mathcal{F}}(S)} v(T), \quad S \subseteq N \ .$$

It is worth noting that for the cases of *coalition structures*, *communication graphs*, *conference structures, partition systems*, *augmenting systems* and *accessible union stable systems,* the decomposition into maximal partitions is *always unique.* That is, if $C_{\mathcal{F}}(S) \neq \{\emptyset\}$ then $C_{\mathcal{F}}(S)$ is made up of a unique partition of a subset of S (see Bilbao, 2003; Proposition 2.9).

Now, in order to make the comparison between γ and ϵ^{Sh} easier, note that γ can also be expressed as the Shapley value of an extended game $v_{\mathcal{F}}$ defined on 2^N as follows:

$$v_{\mathcal{F}}(S) = \sum_{T \in \mathcal{F}(S)} \Delta(T, \mathcal{F}, v), \quad S \subseteq N, \ (N, \mathcal{F}, v) \in RG^N \ .$$

Proposition 2 *$\gamma(N, \mathcal{F}, v) = Sh(N, v_{\mathcal{F}})$, for any $(N, \mathcal{F}, v) \in RG^N$.*

Proof. By definition of $v_{\mathcal{F}}$ and (3) it is immediate that $\Delta(S, 2^N, v_{\mathcal{F}}) = 0$ for all $S \notin \mathcal{F}$, and $\Delta(S, \mathcal{F}, v) = \Delta(S, 2^N, v_{\mathcal{F}})$ for all $S \in \mathcal{F}$. Hence, by Definition (1) we find that $Sh(N, v_{\mathcal{F}}) = \gamma(N, 2^N, v_{\mathcal{F}}) = \gamma(N, \mathcal{F}, v)$. ■

Both definitions, $v_{\mathcal{F}}$ and $\bar{v}_{\mathcal{F}}$, are equivalent when $C_{\mathcal{F}}(S)$ consists of a unique partition of a subset of S and hence, $\epsilon^{Sh} \equiv \gamma$.

Proposition 3 *Let $\mathcal{F}$ be a set system such that if $C_{\mathcal{F}}(S) \neq \{\emptyset\}$ then $C_{\mathcal{F}}(S)$ is made up of a unique partition of a subset of S, for all $S \subseteq N$. Then it holds that*

$$\sum_{T \in C_{\mathcal{F}}(S)} v(T) = \sum_{T \in \mathcal{F}(S)} \Delta(T, \mathcal{F}, v), \quad S \subseteq N \ .$$

Proof. For each $T \in C_{\mathcal{F}}(S)$, using the definition of the dividends, we have that

$$\sum_{K \in \mathcal{F}(T)} \Delta(K, \mathcal{F}, v) = v(T).$$

Thus, since the elements of $C_{\mathcal{F}}(S)$ are disjoints,

$$\sum_{T \in \mathcal{F}(S)} \Delta(T, \mathcal{F}, v) = \sum_{T \in C_{\mathcal{F}}(S)} \sum_{K \in \mathcal{F}(T)} \Delta(K, \mathcal{F}, v) = \sum_{T \in C_{\mathcal{F}}(S)} v(T).$$

■

For more general set systems this equivalence is no longer true, as we will see in the next example.

Example 1 Consider the set system $\mathcal{F} = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}\}$, which corresponds to the case where players 2 and 3 are incompatible, i.e. they cannot be in a coalition at the same time. Here, $C_{\mathcal{F}}(\{1,2,3\}) = \{\{1,2\}, \{1,3\}\}$, and then we have

$$\bar{v}_{\mathcal{F}}(\{1,2,3\}) = \sum_{T \in C_{\mathcal{F}}(\{1,2,3\})} v(T) = v(\{1,2\}) + v(\{1,3\}) \ .$$

On the other hand,

$$\begin{aligned} v_{\mathcal{F}}(\{1,2,3\}) &= \sum_{T \in \mathcal{F}(\{1,2,3\})} \Delta(T, \mathcal{F}, v) \\ &= \Delta(\{1\}, \mathcal{F}, v) + \Delta(\{2\}, \mathcal{F}, v) + \Delta(\{3\}, \mathcal{F}, v) + \Delta(\{1,2\}, \mathcal{F}, v) + \Delta(\{1,3\}, \mathcal{F}, v) \\ &= v(\{1\}) + v(\{2\}) + v(\{3\}) \\ &\quad + [v(\{1,2\}) - v(\{1\}) - v(\{2\})] + [v(\{1,3\}) - v(\{1\}) - v(\{3\})] \\ &= v(\{1,2\}) + v(\{1,3\}) - v(\{1\}) \ . \end{aligned}$$

Hence, we have that $\bar{v}_{\mathcal{F}}(\{1,2,3\}) \neq v_{\mathcal{F}}(\{1,2,3\})$.

Another coincidence lies for the particular case of *union closed systems,* considered in van den Brink *et al.* (2011), where $\mathcal{F}$ is closed under union. There, the set $\sigma(S)$ is defined by

$$\sigma(S) = \bigcup_{T \in \mathcal{F}(S)} T \ ,$$

and the extended game $v_{\mathcal{F}}^{un}$ on G^N is defined by $v_{\mathcal{F}}^{un}(S) := v(\sigma(S))$, for all $S \subseteq N$. The *union value* ζ^{un} is defined by $\zeta^{un}(N, \mathcal{F}, v) = Sh(N, v_{\mathcal{F}}^{un})$, for any union closed system $(N, \mathcal{F}, v)$. Note that when $\mathcal{F}$ is closed under union it holds that $\sigma(S) \in \mathcal{F}(S)$ and $C_{\mathcal{F}}(S) = \{\sigma(S)\}$. In that case we have again that $\sum_{T \in \mathcal{F}(S)} \Delta(T, \mathcal{F}, v) = v(\sigma(S))$ and hence, $v_{\mathcal{F}}^{un}(S) = v_{\mathcal{F}}(S)$ for all $S \subseteq N$. Therefore, trivially $\zeta^{un} \equiv \gamma$ on union closed systems.

4.1.2 The $\Omega(\mathcal{F}, N)$ restriction

In this approach, the set of feasible orders is restricted to those which are compatible with the set system $\mathcal{F}$. Therefore, the value is an expectation of the marginal contributions in each order, being all feasible orders equally likely.

A set system $\mathcal{F}$ can induce a restriction into the set of orders $\Omega(N)$ because it is possible that, for some player i and order ω, the coalition $\{i\} \cup P_{\omega}^{i}$ is not feasible. We denote by $\Omega(\mathcal{F},N)$ the set of orders on N that are compatible with the set system $\mathcal{F}$, that is, $\omega \in \Omega(\mathcal{F},N)$ if and only if $\{i \in N : \omega(i) = r\} \cup P_\omega^i \in \mathcal{F}$ for all $r = 1, ..., n$.

Definition 3 *The value extension* ξ *on* RG^N *is defined by*

$$\xi_i(N,\mathcal{F},v) := \frac{1}{|\Omega(\mathcal{F},N)|} \sum_{\omega \in \Omega(\mathcal{F},N)} m_i^\omega(N,v), \quad i \in N \ . \tag{6}$$

Formula (6) is used in Faigle and Kern (1992) for the case of precedence constrains, in Bilbao and Edelman (2000) for convex geometries, Loehman and Whinston (1976), Loehman, *et al.* (1979), Bilbao and Ordoñez (2008) for augmenting systems, and in Lange and Grabisch (2009) for regular set systems. In Algaba *et al.* (2015) this value is called the precedence Shapley value. Álvarez-Mozos *et al.* (2013) consider the case in which players are political parties ordered in a line, and this formula is called the spectrum value.

All these structures fall into the class of set systems for which (6) is well defined, since $\Omega(\mathcal{F},N) \neq \emptyset$.

We find that this approach is problematic in at least two aspects. First, note that the condition $\Omega(\mathcal{F},N) \neq \emptyset$ fails to consider some interesting classes of set systems. These include the following:

- *Incompatibilities.* Some players are incompatible and then all coalitions which include both of them, and in particular, the grand coalition N, are forbidden[2]. For example, suppose that we want to apply the value as a power index in a weighted majority game that comes from the seats obtained by political parties in a Parliament. And we know that some ideological incompatibilities between two particular parties would make any coalition containing them highly unfeasible. In that case we cannot use ξ as a power index.

- *Pure bargaining problems.* Only the grand coalition can reach feasible agreements. This situation can be expressed by the set system $\mathcal{F} = \{\emptyset, N, \{i\} : i \in N\}$, where $v(N) > \sum_{i \in N} v(i)$.

- *Teams.* There is a minimum number of agents for which a job can be done; or the number of players of feasible coalitions is fixed, as is the case for sport teams.

[2] This is a particular case of a *matroid.*

In all these cases, formula (6) cannot be applied, because for any order ω there are coalition sizes for which there are not feasible coalitions, and then it is impossible to complete any feasible chain.

Second, we find that there are some examples in which the payoffs obtained are rather counterintuitive, despite the fact that ξ is well defined.

Example 2. Consider the game (N, v), where $N = \{1, 2, 3\}$, and v is defined in 2^N by

$$v(1) = 1,\ v(2) = v(3) = v(\{2, 3\}) = 0\ ,$$
$$v(\{1, 2\}) = v(\{1, 3\}) = 3,\ v(\{1, 2, 3\}) = 4\ .$$

Here we can interpret that player *1* is an expert worker, players *2* and *3* are apprentices, and any productive job must be done with the help of the expert player. If we compute the value under the full cooperation setting we obtain

$$Sh_1(N, v) = \frac{8}{3},\ Sh_2(N, v) = Sh_3(N, v) = \frac{2}{3}\ .$$

Suppose now that a market regulator wants to avoid swindles, and for that reason only coalitions that contain the expert player *1* are considered as lawful. With such legal restrictions, the new set system is given by $\mathcal{F} = \{\emptyset, \{1\}, \{1, 2\}, \{1, 3\}, \{1, 2, 3\}\}$. This implies that now we have only two orders compatible with $\mathcal{F}$, that is

$$\begin{aligned} \omega &= (\omega(1) = 1, \omega(2) = 2, \omega(3) = 3)\ , \text{ and} \\ \omega' &= (\omega'(1) = 1, \omega'(2) = 3, \omega'(3) = 2)\ . \end{aligned}$$

If we compute the value extension ξ, we obtain

$$\xi_1(N, \mathcal{F}, v) = 1,\ \xi_2(N, \mathcal{F}, v) = \xi_3(N, \mathcal{F}, v) = \frac{3}{2}\ .$$

This assigns the worst payoff to the expert player. This situation is in complete contradiction with the intuitive idea that it is the expert player who is in the best bargaining position, due to the fact that he is decisive for any productive job.

This is a general behavior for the ξ extension: take any monotonic game (N, v) such that for some $S \subset N$, it holds that $v(T) = 0$ for all $T \subseteq S$. Here players in S are only productive when they are joined with players from outside S. If we build now a new set system $\mathcal{F}$ by deleting all the coalitions in S, i.e. $\mathcal{F} = 2^N \setminus \{T : T \subseteq S\}$, it turns out that with the ξ value extension the payoffs of players in S *increase* and that of the $N\backslash S$ *decrease*, i.e. $Sh_i(N,v) \leq \xi_i(\mathcal{F},N,v)$, for all $i \in S$, and $Sh_j(N,v) \geq \xi_j(N,\mathcal{F},v)$, for all $j \in N\backslash S$. This happens because in $\Omega(\mathcal{F},N)$ we delete all orders for which players of S enter before players of $N\backslash S$, and, in all these orders, the marginal contributions of players in S are zero. This is the reason why the ξ extension always benefits players in S.

Example 3. Consider a river running through three regions. The regions can build harbors to utilize this cheap transportation possibility. Given financial restrictions, each region can build at most one harbor. Neighboring regions might join efforts to build a common harbor and thus save costs. We include the option to share a common harbor for the three regions. The only restriction is that the two extreme regions, which are separated by the central one, cannot build a common harbor that serves only the two of them. Denote the set of players by $N = \{1, 2, 3\}$, and let 2 be the central region. The saving function v has positive value only when players share a harbor, and suppose the savings are

$$v(1) = v(2) = v(3) = 0\ ,$$
$$v(\{1,2\}) = v(\{2,3\}) = 2,\ v(\{1,3\}) = 0,\ v(\{1,2,3\}) = 7\ .$$

The value under the full cooperation setting is

$$Sh_1(N,v) = 2,\ Sh_2(N,v) = 3,\ Sh_3(N,v) = 2\ .$$

Region 2 benefits from its central position in the line. Nevertheless, as only coalition $\{1,3\}$ is forbidden, the associated set system is $\mathcal{F} = 2^N\backslash\{1,3\}$. As players are ordered in a line (or spectrum), we have two orders incompatible with $\mathcal{F}$, which corresponds to the case when the central player enters last in the order, that is, the orders $\omega = (\omega(1) = 1, \omega(2) = 3, \omega(3) = 2)$ and $\omega' = (\omega'(1) = 2, \omega'(2) = 3, \omega'(3) = 1)$ are excluded. The value extension ξ gives the values

$$\xi_1(N,\mathcal{F},v) = 3,\ \xi_2(N,\mathcal{F},v) = 1,\ \xi_3(N,\mathcal{F},v) = 3\ .$$

In this setting, these values coincide with those obtained in Loehman and Whinston (1976), Loehman *et al.* (1979), and in the spectrum value of Álvarez-Mozos *et al.* (2013). We find again the same behavior of ξ: the reduction of the players' cooperation possibilities increases their bargaining power.

On the contrary, it is an easy exercise to check that in examples, 2 and 3, our value extension coincides with the Shapley value in the original game (N, v), i.e. $\gamma(N, \mathcal{F}, v) = Sh(N, v)$.

4.2 Stability and the inessential coalition out property

Myerson (1977, 1980) also considered the question of whether it is beneficial for the members of a coalition S to be feasible:

Stability: $\psi_i(N, \mathcal{F}, v) \geq \psi_i(N, \mathcal{F} \backslash S, v)$ for all $i \in S$ and $S \in \mathcal{F}$.

If the game (N, v) is superadditive, the property holds for graph restricted games and conferences structures, and it was also proved[3] for the case of union stable systems in Algaba *et al.* (2001) (see Proposition 3.3). As we have seen before, our value γ coincides with the value extension considered for graph restricted games, conferences structures and union stable systems; hence it shares the same properties.

However, for more arbitrary set systems this property does not necessary hold for the value γ.

Example 4. Consider the set system

$$\mathcal{F} = \{\emptyset, \{1\}, \{1, 2\}, \{1, 2, 3\}, \{1, 2, 4\}, \{1, 2, 5\}, \{1, 2, 3, 4, 5\}\},$$

and coalition $S = \{1, 2\}$. The reader can check that

$$\gamma_1(N, \mathcal{F}, v) - \gamma_1(N, \mathcal{F} \backslash S, v) = -\frac{1}{6}\Delta(\{1, 2\}, \mathcal{F}, v) = -\frac{1}{6}\left(v(\{1, 2\}) - v(1)\right).$$

Therefore, we have $\gamma_1(N, \mathcal{F}, v) < \gamma_1(N, \mathcal{F} \backslash S, v)$ when $v(1) < v(\{1, 2\})$.

We recall that a dividend $\Delta(S, \mathcal{F}, v)$ measures the additional gains that players obtain by cooperation within the coalition S. The above example shows that, even if game $(N, \mathcal{F}, v)$ all of its dividends are positive, it is not a trivial task to decide whether it is beneficial or not for its members that this coalition be feasible, even if the original game v is convex.

In general, by means of a simple induction argument, it is easy to prove that:

[3] Actually the property considered is called *basis monotonicity*. The property imposes some additional restrictions to the feasible set that can be considered.

Proposition 4 *For any* $(N, \mathcal{F}, v) \in RG^N$ *and* $S \in \mathcal{F}$*, it holds that, for each* $T \in \mathcal{F}$ *with* $T \supset S$*, there exists* $k_T \in \mathbb{R}$ *such that*

$$\Delta(T, \mathcal{F}, v) - \Delta(T, \mathcal{F} \backslash S, v) = k_T \Delta(S, \mathcal{F}, v) \ .$$

From this fact, it is immediate the following corollary:

Corollary 1 *For any* $(N, \mathcal{F}, v) \in RG^N$ *and* $S \in \mathcal{F}$*, it holds that there exists* $\lambda_S \in \mathbb{R}$ *such that*

$$\gamma_i(N, \mathcal{F}, v) - \gamma_i(N, \mathcal{F} \backslash S, v) = \lambda_S \Delta(S, \mathcal{F}, v), \ i \in S \ .$$

The sign of λ_S will depend on $\mathcal{F}$ and there is not a simple formula for λ_S with an arbitrary set system.

Nevertheless, a middle requirement that we can impose on RG^N is the following: if the feasibility of a coalition S does not sum additional gains with respect to gains obtained previously without it, then the deletion of this coalition from $\mathcal{F}$ should not affect the final payoffs.

Inessential coalition out: For any $(N, \mathcal{F}, v) \in RG^N$ and $S \in \mathcal{F}$, $\psi(N, \mathcal{F}, v) = \psi(N, \mathcal{F} \backslash S, v)$ when $\Delta(S, \mathcal{F}, v) = 0$.

This is a dynamic property that specifies when we can add or delete coalitions into the set system $\mathcal{F}$, without altering the value. A similar property was introduced in van den Brink *et al.* (2011):

Independence of irrelevant coalitions: Let two games $v, w \in G^N$ such that $v(S) = w(S)$ for all $S \in \mathcal{F}$, then $\psi(N, \mathcal{F}, v) = \psi(N, \mathcal{F}, w)$.

This property is static, because in it $\mathcal{F}$ is fixed, and it specifies when we can change the worth of the game without altering the final payoffs.

Proposition 5 γ *satisfies the properties of* inessential coalition out *and* independence of irrelevant coalitions *in RG.*

Proof. Corollary (1) implies that γ satisfies the inessential coalition out property. Let two games $v, w \in G^N$ such that $v(S) = w(S)$ for all $S \in \mathcal{F}$, then we have that $\Delta(S, \mathcal{F}, v) = \Delta(S, \mathcal{F}, w)$ for all $S \in \mathcal{F}$, and by definition of γ it holds that $\gamma(N, \mathcal{F}, v) = \gamma(N, \mathcal{F}, w)$; therefore γ satisfies independence of irrelevant coalitions. ■

4.3 Value extensions on RG^N

Note that a minimal requirement that a solution ϵ^φ defined on RG should satisfy, in order to be considered as an *extension* of a solution φ defined on G, is that $\epsilon^\varphi\left(N, 2^N, v\right) = \varphi(N, v)$ for all $(N, v) \in G^N$. We will see that if we impose the properties of *inessential coalitions out* and *independence of irrelevant coalitions*, we find that the way in which such extensions can be done is fixed. This is the content of the next proposition.

Proposition 6 *Let a value ϵ^φ on RG be an extension of a value φ on G, then ϵ^φ satisfies* independence of irrelevant coalitions *and* inessential coalition out *properties if and only if $\epsilon^\varphi(N, \mathcal{F}, v) = \varphi(N, v_\mathcal{F})$, where $v_\mathcal{F} \in G^N$ is defined by*

$$v_\mathcal{F}(S) = \sum_{T \in \mathcal{F}(S)} \Delta(T, \mathcal{F}, v), \quad S \subseteq N \ .$$

Proof. Let $(N, \mathcal{F}, v) \in RG^N$ and let ϵ^φ be an extension of φ (i.e. $\epsilon^\varphi\left(N, 2^N, v\right) = \varphi(N, v)$). By construction, we have that $\Delta\left(S, 2^N, v_\mathcal{F}\right) = 0$ for all $S \notin \mathcal{F}$. Therefore, the inessential coalition out property implies that $\epsilon^\varphi\left(N, 2^N, v_\mathcal{F}\right) = \epsilon^\varphi\left(N, 2^N \backslash S, v_\mathcal{F}\right)$ for all $S \notin \mathcal{F}$. Applying this argument repeatedly, deleting coalitions from $2^N \backslash \mathcal{F}$, we find that $\epsilon^\varphi(N, \mathcal{F}, v_\mathcal{F}) = \epsilon^\varphi\left(N, 2^N, v_\mathcal{F}\right)$. But note that $v_\mathcal{F}(S) = v(S)$ for all $S \in \mathcal{F}$ and hence, the property of independence of irrelevant coalitions implies that $\epsilon^\varphi(N, \mathcal{F}, v_\mathcal{F}) = \epsilon^\varphi(N, \mathcal{F}, v)$. Finally, we obtain that

$$\epsilon^\varphi(N, \mathcal{F}, v) = \epsilon^\varphi(N, \mathcal{F}, v_\mathcal{F}) = \epsilon^\varphi\left(N, 2^N, v_\mathcal{F}\right) = \varphi(N, v_\mathcal{F}) \ .$$

Reciprocally, let ϵ^φ be an extension of φ such that $\epsilon^\varphi(N, \mathcal{F}, v) = \varphi(N, v_\mathcal{F})$ for any $(N, \mathcal{F}, v) \in RG^N$. Let $(N, \mathcal{F}, v) \in RG^N$ and $S \in \mathcal{F}$ with $\Delta(S, \mathcal{F}, v) = 0$. In such case, by construction, $(N, v_\mathcal{F}) = \left(N, v_{\mathcal{F} \backslash S}\right)$ and then

$$\epsilon^\varphi(N, \mathcal{F}, v) = \varphi(N, v_\mathcal{F}) = \varphi\left(N, v_{\mathcal{F} \backslash S}\right) = \epsilon^\varphi(N, \mathcal{F} \backslash S, v) \ .$$

Hence, ϵ^φ satisfies the inessential coalitions out property.

Now let two games $(N, \mathcal{F}, v), (N, \mathcal{F}, w) \in RG^N$ with $v(S) = w(S)$ for all $S \in \mathcal{F}$. Again, by construction, $(N, v_\mathcal{F}) = (N, w_\mathcal{F})$ and then

$$\epsilon^\varphi(N, \mathcal{F}, v) = \varphi(N, v_\mathcal{F}) = \varphi(N, w_\mathcal{F}) = \epsilon^\varphi(N, \mathcal{F}, w) \ .$$

Hence, ϵ^φ satisfies independence of irrelevant coalitions. ■

As an example, we illustrate how to define two well-known families of solutions in RG^N: the selectope, also called the Harsanyi set, and the family of efficient, linear and symmetric values.

A sharing system is a collection of weights which specifies how to share the dividends of a game. If we denote by $P(N)$ the family of such sharing systems in N, $P(N)$ is defined by

$$P(N) = \left\{ p = \left(\left(p_i^S\right)_{i \in S} \right)_{S \subseteq N} : p_i^S \geq 0 \text{ for all } i \in S \text{ and } \sum_{i \in S} p_i^S = 1, \text{ for each } S \subseteq N \right\} .$$

For a game (N, v) and a sharing system $p \in P(N)$, let the payoff vector $\phi(p)(N, v) \in \mathbb{R}^N$ be given by

$$\phi(p)_i (N, v) = \sum_{S \subseteq N:\ i \in S} p_i^S \Delta(S, v), \quad i \in N \ .$$

We call $\phi(p)(N, v)$ the *Harsanyi payoff vector*. Let $H(N, v)$ be the set of all Harsanyi payoff vectors of the game (N, v), that is

$$H(N, v) = \{\phi(p)(N, v) :\ p \in P(N)\} \ .$$

This set was introduced in Hammer et al. (1977) with the name of *selectope*, and independently in Vasil'ev (1978, 1981). See also Derks *et al.* (2000) and Derks *et al.* (2010).

Now given a set system $\mathcal{F}$ and its associated feasible dividends collection $\Delta(\mathcal{F}, v) = (\Delta(S, \mathcal{F}, v))_{S \in \mathcal{F}}$ any $\phi(p)$ can be extended onto RG^N as follows:

$$\epsilon_i^{\phi(p)}(N, \mathcal{F}, v) = \sum_{S \in \mathcal{F}(i,N)} p_i^S \Delta(S, \mathcal{F}, v), \quad i \in N \ .$$

Note that $\Delta(S, v_{\mathcal{F}}) = 0$ for all $S \notin \mathcal{F}$. Therefore, we can also define $\epsilon^{\phi(p)}$ by $\epsilon^{\phi(p)}(N, \mathcal{F}, v) = \phi(p)(N, v_{\mathcal{F}})$.

Another well-known family considered in the literature is that of the efficient, linear and symmetric values (ELS-values). There are several alternative ways to express such ELS-values. For example, in Razdik and Driessen (2013) a value ϕ is an ELS-value if there exists a unique collection of constants $\{b_s |\ s = 0, 1, 2, ..., n\}$, with $b_0 = 0$ and $b_n = 1$ such that $\phi(N, v)$ is of the form

$$\phi_i(N, v) = \sum_{S \subseteq N \setminus i} \frac{s!\,(n - s - 1)!}{n!} \left[b_{s+1} v\,(S \cup i) - b_s v(S)\right], \quad i \in N \ .$$

Alternative expressions can also be found in Ruiz *et al.* (1998), Hernandez-Lamoneda *et al.* (2008), and Chameni and Andjiga (2008).

With the help of any of these representations, we can extend any ELS-value $\phi(N, v)$ on RG^N by $\epsilon^{\phi}(N, \mathcal{F}, v) = \phi(N, v_{\mathcal{F}})$.

In both cases, the Harsanyi set and the ELS-values, are extended in such a way that the *independence of irrelevant coalitions* and *the inessential coalition out* properties are preserved.

Note that this general construction can also include solutions other than the previously mentioned as, for example, the Core, the Least-Core, the Kernel, the Nucleolus and the family of semivalues.

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