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Local coordination and global congestion in random networks☆

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Abstract

This paper analyzes the impact of local and global interactions on individuals' action choices. Players are located in a network and interact with each other with perfect knowledge of their neighborhood and probabilistic knowledge of the complete network topology. Each player chooses an action, from some finite set, which imposes an externality on their neighbors as well as an externality on the complete network. Players deal with two opposing forces: they obtain utility from sharing their choices with their neighbors (positive local externality) but suffer disutility from sharing the same choice with all members of the network (negative global externality). Economic and social phenomena exhibiting these features are: the adoption of cost-reducing innovations, clusters of firms, time schedule choices, the adoption of subcultures and fads, among others.

We find the conditions for the existence of all symmetric Bayesian Nash equilibria and translate them to a characterization in terms of the main properties of the network topology. The balance between local satisfaction and global dissatisfaction partially explains the equilibrium outcome. The players who finally decide on the type of equilibria are those that are either highly connected (hubs) or poorly connected (peripherals) to the others. On the one hand, hubs try to coordinate their action choices and on the other, peripherals are only worried about congestion and play the least selected actions of the network. Some examples illustrate our main results. As a by-product we also show the failure of symmetric equilibrium existence for some congestion costs when the players' types are finite.

Keywords: Random Network, Externalities, Action Selection, Bayesian Nash equilibria.
JEL Classification: C72, D71, D85, H40, R41.

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1. Introduction

Many social and economic activities exhibit local externalities but, at the same time, suffer from the inability of agents to coordinate their actions which, in turn, is a source of congestion in societies. For instance, individuals following the same schedule have the opportunity to share common time with others, which is highly valuable for them: colleagues in a firm can meet in their coffee break, friends can see each other after work to speak and have a drink, relatives can get together in the evening on working days or at the weekend. But these individuals access public services such as public transport, highways, sport facilities, supermarkets, cinemas, airports, etc., which are usually congested at rush hours because of the regularity of society's schedules. As a result, individuals suffer the inconvenience of sharing public services with many other people (global negative externality or congestion) in order to share common time with colleagues, friends and relatives (positive local externality). This effect has been called the *tragedy of the commons* in the analysis of air traffic congestion (Mayer and Sinai [20]), but can be extended to many other contexts where there are multiple agents who do not take into account the externality they create for others.

For example, firms often benefit when their business partners (suppliers, firms of complement goods) adopt cost-reducing innovations if this set of partners is a small subset of the total set of potential adopters of these innovations. But too many adopters (substitute firms) may give rise to a negative externality. Similarly, firms' choice of location may suffer from the same coordination problem: firms may like to locate in clusters with other firms in order to obtain increasing returns from sharing local indivisible facilities (or a common local public good). However, too many firms in the same cluster may create a congestion problem and reduce the initial advantage of being together.

The adoption of *subcultures*, social groups with particular behaviors or beliefs, by youth can also be explained by our model. A young person will adopt a certain subculture if the proportion of their friends following it is big enough. However, young people like belonging to something unique and exclusive, thus the attractiveness of a subculture decreases with the proportion of people in society following it. In like fashion, the adoption of fads where the exclusivity is part of their attractiveness can also be approached by our model.

Coordination failure, or agents' uncertainty about the action of other agents, may be an important source of congestion in large decentralized societies. In the *El Farol* or Santa Fe bar problem, Arthur [2] provides a simple paradigm for congestion and coordination problems that may arise in societies. *El Farol* is a bar in Santa Fe (Argentina). The bar is popular, but becomes overcrowded when more than sixty people over one hundred attend on any given evening. Everyone enjoys themselves when fewer than sixty people go, but no one has a good time when the bar is overcrowded. The *El Farol* problem emphasizes the difficulty of coordinating the actions of independent agents without a centralized mechanism. Unlike the standard public good framework, in this scenario fully informed optimizing agents will not increase consumption of a publicly available resource until it experiences an inefficient level of congestion. If agents could predict the behavior of other agents perfectly the bar would never be crowded and all the patrons would have a good time. The only source of congestion, at least in a deterministic framework, is the inability of agents to coordinate their actions. Although the *El Farol* problem initially explored the collective dynamics of boundedly rational agents, it is also interesting as a simple model of congestion and coordination behavior that occurs with shared resources like Internet bandwidth. Arthur [2] believed that any solution to the *El Farol* problem would require heterogeneous agents, that is, agents who pursue different strategies. A related problem is the one of Challet and Zhang [8], who proposed an alternative version of the *El Farol* problem, known as the *Minority Game*: An odd number of players have to choose simultaneously one of two rooms. The players who choose the less crowded room receive a reward of one euro, the others receive nothing. In general, this class of game is interesting when several agents must take decentralized decisions on whether to access a scarce resource, knowing that at most a fixed number of them will be able to enjoy its benefits. A minority game is basically a repeated coordination game. Renault *et al.* [25] were the first to consider a minority game from the traditional strategic viewpoint of repeated games with public monitoring. Renault *et al.* [26] extend the analysis to public signals but private strategies, namely, they depend on the public signal and on the private history of each player.

We analyze the above insights in a static model where individuals enjoy being at the bar with their friends or relatives but suffer from the congestion created by all the agents in society. We will assume that congestion is an increasing function on the number of individuals choosing the same action, unlike the *El Farol* problem where there is only congestion when the proportion of individuals in the bar is above a certain threshold. Thus, we consider that the congestion cost is not an *all or nothing* concept but a non-linear continuous function on the proportion of individuals making the same choice. Namely, when the number of players choosing the same is small, then the addition of a new player with the same choice will not increment the congestion cost substantially, while if the number of players following the same action is large, then the new player will cause a large increment in the congestion.

The relationships between individuals are modeled as a network, where each node is an individual and any (undirected) link between two nodes represents some kind of relationship between them such as friendship, family ties, firms of complement goods, etc. Each individual only has a local knowledge of the network, namely, they know their neighbors (to whom they are linked to) but they do not know who their neighbors' neighbors are. This lack of information about the network's topology is modeled by considering it as an instance of a random network where individuals know the degree probability distribution over the nodes of the network.

Individuals simultaneously choose their actions from a finite set, which imposes an externality on their neighbors as well as an externality on the complete network, and then obtain an utility. The optimal (Bayesian Nash) decision taken by each individual depends on the spread of their connections in the network (their degree), on the degree probability distribution, and on the balance between the local and global externalities which impact on their utility. It is assumed that each individual's utility depends positively on the proportion of neighbors choosing the same action (positive local externality) and negatively on the proportion of the network members doing the same, because it creates congestion (negative global externality).

Our contribution is twofold. We find first the conditions for the existence of all *sym-*

metric Bayesian Nash equilibria and discus the existence failure when the support of the players' degrees is finite. Next, we translate the condition on equilibrium existence to an equilibrium characterization in terms of the main properties of the network topology. The balance between local satisfaction and global dissatisfaction partially explains the equilibrium outcome. The players who finally decide the type of equilibria are those that are either highly connected (hubs) or poorly connected (peripherals) to the others. On the one hand, hubs try to coordinate their action choices and choose the same action as long as congestion is not very high. On the other, peripherals are only worried about congestion and play the least selected actions of the network.

To motivate our analysis notice that a rough calculation would give us two possible (Bayesian Nash) equilibrium solutions: one where all individuals choose the same action (homogeneous pure profile), and another where each individual chooses their actions randomly giving all options the same probability (uniformly mixed profile). Intuitively, homogeneous pure profiles could be equilibrium outcomes when the positive local externality dominates the negative global externality, whereas uniformly mixed profiles would be equilibrium solutions otherwise.

However, common intuition needs to be polished since both local and global network properties play an important role in equilibrium choices. This comes from the observation that the network topology defines two important features such as *hub players* (highly connected nodes) and *peripheral players* (poorly connected nodes). Although each individual's value function will depend on both the average action profile followed by the network and the average action profile of their neighbors, their relative weight will depend on the individual's number of connections. Thus, the network average action profile is particularly important for peripherals because, by definition, their number of neighbors is very small and therefore their choice will mostly be driven by the network global topology. A peripheral player is only worried about congestion and to reduce it as much as possible she choose the least frequent action. Thus, when peripherals are frequent homogeneous pure equilibrium profiles are impossible to sustain. On the contrary, the hubs or highly connected players' choices will mainly depend on the average profile of their neighbors' actions, i.e. on the network local properties. It's very likely that hubs will be linked to other hubs and try to coordinate their choices to maximize their utility. Thus it is expected that peripherals play mixed strategies while hubs choose pure strategies, that is a hybrid equilibrium. However, when the maximum degree of the network is bounded and the proportion of hubs is too high, then the congestion disutility may prevail and make it difficult to guarantee symmetric equilibrium profiles. Therefore, Bayesian Nash Equilibria are expressed in terms of the proportion of hubs and peripherals which, in turn, is given by the weight of the tails of the degree probability distribution. To the best of our knowledge this is the first time that both local and global effects are analyzed from the network perspective. We finish this section with a review of related literature.

1.1. Related literature

Our paper is a contribution to network games, an active area of research over the last few years. A complete review of this literature exceeds the intention of this section, so we refer readers to the extensive overview in Goyal [16], Jackson [18] and more recently Galeotti *et al.* [13].

We assume that the network of relationships between individuals is fixed. When an individual chooses their action they obtain utility from sharing their choice with their neighbors but suffer the disutility of sharing the same choice with all members of the network. Thus, an individual's net utility depends both on their action as well as on other individuals' actions, forcing them to play in a strategic way. The focus is on large networks, where a change in the behavior of one individual could drastically modify their utility, while having a marginal impact on other individuals' utilities. Thus, we will consider that there are an infinite number of individuals in the network as in Galeotti and Vega-Redondo [14] and Morris and Shin [21]. This assumption makes the computations easier and does not have any effect on the main results.

We consider a random network, where individuals do not know the complete topology of the network but rather the degree distribution (which is fixed). Random networks were first used in Newman [22]. An overview of this literature can be found in Newman *et al.* [23] and some applications of these models to economic problems in Ioannides [17]. More recently Galeotti and Vega-Redondo [14] use random networks to study how local externalities shape agents' strategic behavior when the underlying network is volatile and complex. Galeotti *et al.* [13] and Sundararajan [28] analyze local networks, where each individual's utility depends on their own action as well as on their neighbors' actions. The first paper provides a framework with random networks to characterize the behavior and payoffs of individuals according to different factors in the model. The second paper presents a model where individuals value the adoption of a product by a heterogeneous subset of other individuals in their neighborhood, and have incomplete information about the structure and strength of adoption complementarities among all other individuals.

Recently several authors have become interested in *local networks*, where a player's payoff depends only on their own actions and those of their neighbors. Galeotti *et al.* [13] provide a general model and analyze how a given individual's behavior is affected by their position within the network and the nature of the game (strategic substitutes versus complements and positive versus negative externalities, and the level of information.) Sundararajan [28] presents a model of local effect for the analysis of an adoption game. Among other results, these articles show the existence of equilibria in pure strategies and give some properties that they verify. Our model differs in that we consider a payoff function which has both local and global externalities, but we use specific functional forms close to the ones found in Galeotti and Vega-Redondo [14] and Ballester *et al.* [3]. Those papers, however, analyze local games and a continuum of players' space of actions.

Following the above approach, Chen and Gostoli [9] embed the *El Farol* problem in a *social network* connecting the agents and through which the agents can access the information regarding their neighbors' choices and strategies. In the original set up of Arthur [2] the agents base their decisions on *global information*, represented by the bar aggregate attendance, a feature that is likely to cause herding behavior, making it very difficult to coordinate their activities. Chen and Gostoli go a step further and analyze whether coordination will be improved if, instead, the agents make use of local information, represented by the attendance of their closest neighbors. They simulate the bar attendance dynamics though cellular automata. Our set up could be seen as a static mixed majority/minority game (see for example, Marsili [19], De Martino *et al.* [10] and Cannings [7]) where players, instead of following a dynamic rule to choose actions, are embedded in a random social network which determines their utility and hence their action choice. Hub players will play the majority game and would like to coordinate in the same action as long as congestion is not very high. Peripherals will play a minority game and would like to play the least chosen action by the majority, even when congestion is moderate. The equilibrium that results, either coordination in one action, mixed equilibrium or hybrid equilibrium where some players choose the same action and the others play a mixed action will depend on both the weight of the tales of the players' degree probability distribution and on the balance between the positive local externality and the negative global externality.

This paper is organized into seven sections. Section 2 provides the general framework. Section 3 presents the main results on equilibrium existence and Section 4 characterizes equilibria in terms of the network topology. Section 5 illustrates the results of Sections 4 in Scale free and Poisson random networks. Some comparative statics for two-type players' examples are offered in Section 6. Finally, Section 7 concludes the paper.

2. The model

There is a countable infinity of agents (players) N, and A is a finite set of actions for them. We assume that there are only two possible actions, [1] i.e. $A = \{m, e\}$. For each player $i \in N$, we denote their action by $a_i \in A$. What is relevant in the analysis is that if only one individual changes their decision, then the other individuals' payoffs do not change (or change only marginally). Thus, the analysis could be carried out by considering a large number of individuals and the main results will not be affected.

Non-directed graphs are used to model network relationships between individuals. In such graphs the nodes correspond to the agents and the links correspond to the bilateral relationships between them. Let g be such a network.

Each individual $i \in N$ has a number of relationships with other agents in g that defines their set of neighbors, N_i, and their degree, k_i (the cardinality of N_i). Each agent knows their degree but does not know the degree of the other nodes in the network. We assume that players know the degree distribution that is fixed and characterized by the probability distribution

$$\mathbf{p} = \{p_k\}_{k \in K} \tag{1}$$

where K, its support, is a subset (not necessarily finite) of the positive integer numbers, $K \subseteq \mathbb{N}^*$, and p_k denotes the probability of finding a player in the network who has k neighbors. We assume that the first moment of the random variable defined by $\mathbf{p}$ is finite, thus let d be the average degree, i.e. $d = \sum_{l \in K} l p_l$. Notice that isolated players are not

[1] This is a simplification. The model with a finite number of actions extends easily.

allowed since we assume that each individual in society maintains at least one relationship with another.

Players interact with each other as determined by g. The network is equiprobable chosen from among all the possible networks that have a given degree distribution $\mathbf{p}$. Thus, we are assuming that no player knows their neighbors' degree but all of them know the overall degree distribution and the random network is fully characterized by it.

The influence of a player in the network is measured by their centrality. The simplest measure of centrality is a player's degree, which only uses local information and is invariant with respect to the rest of the graph.[2] An individual with high degree is a central player with respect to a local portion of the graph and becomes a *hub* of the network, while another individual with low degree is a *peripheral* in a local portion of the graph. Notice that both hubs and peripherals are relative concepts: they depend on the degree probability distribution of each network. Given a network, an individual's degree should be considered high or low with respect to the average degree of the network she belongs to. Thus, a relative measure of centrality may be helpful to classify a node as either a hub or a peripheral.

Definition 1. *[Relative degree] Given a network g and its degree distribution $\mathbf{p} = \{p_k\}_{k\in K}$, we define the relative degree of a node as the ratio between its degree and the network average degree, i.e. given node $i \in N$, its relative degree is k_i/d, where $d = \sum_{l\in K} lp_l$.*

Relative degree play a central role in the characterization of the equilibria. Nodes with high relative degree will be considered as hubs and nodes with low relative degree will be peripherals.

If a player chooses a neighbor randomly, then they will not know their degree but will know that neighbor's degree distribution. Assuming independence across neighbors' degrees, the probability of arriving at a node is proportional to its degree, and we can compute that

$$\widetilde{p}_k = \frac{kp_k}{\sum_{l\in K} lp_l} \tag{2}$$

is the probability of a node having degree k when it is selected randomly from among a player's neighbors. Let this distribution be denoted by $\widetilde{\mathbf{p}} = \{\widetilde{p}_k\}_{k\in K}$.

Mixed strategies are allowed, so that the decision of any player is an element in $\Delta(A)$, which is the set of all probability distributions over A. Given that A has two elements, we can identify the 1-dimensional simplex $\Delta(A)$ with the interval $[0, 1]$. Therefore, player i's action is $x_i \in [0, 1]$, where x_i is the probability of choosing action m, and then $1 - x_i$ is the probability of choosing action e.

Prior to interaction each player i has to decide their action $x_i \in [0, 1]$ individually and independently of the other players. This decision can only depend on their own degree and the degree distribution on the other players' degrees. Let $\{x_i, \mathbf{x_{-i}}\}$ be the profile of actions,

[2] Other concepts of centrality, such as Closeness, Betweenness and Bonacich' measure, need to know the complete topology of the network to be calculated. A description of these measures can be found in Jackson [18]

where x_i is the action chosen by i and $\mathbf{x_{-i}}$ those of the other players. Let $(x_j)_{j\in N_i}$ be the profile of actions of i's neighbors. We assume that the net payoff function of player i has two components, the gross payoff function, f, which measures local externalities, and the congestion function, h, which measures global externalities:

$$u_i[x_i, \mathbf{x_{-i}}] = f[x_i, (x_j)_{j\in N_i}] - h[x_i, \mathbf{x_{-i}}].$$

Assuming ex-ante symmetry across players, player i's gross payoff depends on their action and the actions of their neighbors,

$$f : \Delta(A) \times \Delta(A)^{N_i} \to \mathbb{R}_+,$$

while the congestion depends on the actions chosen by all the players in the network,

$$h : \Delta(A)^N \to \mathbb{R}_+.$$

2.1. The Bayesian Game

The strategic situation is modeled as a classical Bayesian game, where each player's type is identified with their degree and all the players' types are drawn independently according to the prevailing degree distribution $\mathbf{p}$. Therefore, the type space for every player is K and their beliefs on the other types is the degree distribution $\mathbf{p}$. Each player's strategy is a mapping from their type to set $[0,1]$. In other words, under independence across degrees and as in Galeotti and Vega-Redondo [14], we posit that each player chooses an action induced only by their own degree, the degree distribution $\mathbf{p}$ and their prediction of the other players' actions $\mathbf{x} = \{x_l\}_{l\in N}$ which specifies how every other player anticipates choosing their action, depending on their degree. Thus, we analyze symmetric Bayesian strategies, i.e., all players with the same degree choose the same strategy.

Denote by $v_k[x, \mathbf{x}]$ the expected payoff function of a k-degree player who chooses action x and expects the degree contingent strategy $\mathbf{x} = \{x_l\}_{l\in K}$,

$$v_k[x, \mathbf{x}] = E_{\widetilde{\mathbf{p}}}[f[x, (x_{k_j})_{j\in N_i}]] - E_{\mathbf{p}}[h[x, (x_{k_j})_{j\in N}]].$$

We have defined an incomplete information game where the player's degree defines their type. The first objective of this paper is to study the strategy profiles (indexed by the degree) that are symmetric Bayesian-Nash Equilibrium (BNE).

Definition 2. *[Symmetric Bayesian-Nash Equilibrium] A strategy profile* $\mathbf{x}^* = \{x_k^*\}_{k\in K}$ *is a symmetric* Bayesian-Nash Equilibrium *(BNE) if it satisfies:*

$$x_k^* \in argmax_{x\in[0,1]} v_k[x, \mathbf{x}^*] \tag{3}$$

for all $k \in K$.

A strategy profile is a BNE if no player can deviate unilaterally and benefit from that deviation.

To provide a precise specification of function $v_k[x_k, \mathbf{x}]$ and characterize the BNE we have to consider in detail the two functions that define the expected net payoff function.

2.2. Local externalities: the expected gross payoff function

The gross payoff function, f, measures the utility that a player, say i, obtains by the interaction with their neighbors. We define the *gross payoff function* of a player *as the expected proportion of their neighbors choosing their same action.* Therefore, the local interaction component exhibits positive externalities since the player i's gross payoff increases with the proportion of neighbors choosing the same action than i.

The player i's gross payoff function depends on two random variables, the proportion of their neighbors choosing their same action, and their own action. The first random variable has a distribution governed by $\widetilde{\mathbf{p}}$, and the second follows a Bernoulli distribution with probability x_{k_i}. However, by the law of total probability, the expected proportion of their neighbors choosing their same action is given by the expected proportion of their neighbors choosing action m times the probability of player i's action to be m plus the expected proportion of their neighbors choosing action e times the probability of player i's action to be e:

$$\begin{aligned} E_{\widetilde{\mathbf{p}}} f[x_{k_i}, (x_{k_j})_{j\in N_i}] &= E_{\widetilde{\mathbf{p}}} f[m, (x_{k_j})_{j\in N_i}] Prob(a_i = m) + E_{\widetilde{\mathbf{p}}} f[e, (x_{k_j})_{j\in N_i}] Prob(a_i = e) \\ &= E_{\widetilde{\mathbf{p}}} f[m, (x_{k_j})_{j\in N_i}] x_{k_i} + E_{\widetilde{\mathbf{p}}} f[e, (x_{k_j})_{j\in N_i}](1 - x_{k_i}). \end{aligned}$$

The expression $E_{\widetilde{\mathbf{p}}} f[m, (x_{k_j})_{j\in N_i}]$ is the expected value of a random variable, the proportion of neighbors choosing action m. One way to obtain an explicit specification of this expected value is to make explicit the support of the random variable, compute the probability mass of each element in the support and calculate the expectation.[3] An alternative way is to realize that the expected value of $f[m, (x_{k_j})_{j\in N_i}]$ is equal to the probability that a randomly chosen player i's neighbor has played action m, and compute this probability using, again, the law of total probability:

$$\begin{aligned} E_{\widetilde{\mathbf{p}}} f[m, (x_{k_j})_{j\in N_i}] &= Prob(\text{Randomly chosen neighbor (r.c.n.) plays } m) \\ &= \sum_{l\in K} Prob(\text{r.c.n. plays } m|\text{r.c.n. has degree } l) Prob(\text{r.c.n. has degree } l]) \\ &= \sum_{l\in K} x_l \widetilde{p}_l \end{aligned}$$

[3] This is the way followed in Galeotti and Vega-Redondo [14] to compute a close expression. The support of the random variable is the set of all possible distributions of the degrees of player i's neighbors, i.e., the set of all vectors of integer components such that the sum of the components is equal to k_i. The random variable follows a multinomial distribution. This framework has to assume that players' degrees have a bound from above.

Notice that the last term does not depend on the identity of the player but depends on the player's action which is degree contingent. Let us define the average proportion of neighbors following action m in terms of the distribution of the neighbors' degree as

$$\widetilde{x} = \sum_{l \in K} x_l \widetilde{p}_l. \tag{4}$$

Similarly, we obtain that $E_{\widetilde{\mathbf{p}}} f[e, (x_j)_{j \in N_i}] = 1 - \widetilde{x}$. Thus, the expected gross payoff function of a player with degree k can be written as,

$$E_{\widetilde{\mathbf{p}}} f[x_k, \mathbf{x}] = x_k \widetilde{x} + (1 - x_k)(1 - \widetilde{x}) \tag{5}$$

that is independent of the player's identity. Therefore, the expected gross payoff of k-type player depends on its own action and on the average proportion of neighbors choosing the same action in terms of the distribution of the neighbors' degree.

2.3. Global externalities: the expected congestion function

The congestion function, h, measures a player's dissatisfaction from, for example, the use of a public service simultaneously with individuals in the network that have chosen their same action. Thus, this is a global interaction and exhibits negative externalities, as a player's payoff will be decreasing on the number of players choosing the same action as theirs.

We propose a *congestion function that is quadratic on the expected proportion of players choosing the same action as that player.* This relationship reflects what is commonly seen in real life. When the number of players choosing the same action is small, then the addition of a new player with the same action will not increment congestion substantially, while if the number of players following the same action is large, then the new player will cause a large increment in congestion. This fact is reflected through the quadratic dependence of the congestion on the number of subjects choosing the same action as theirs.

Let us consider a player i with degree k. Similar reasoning as the above for the gross payoff function allows us to write the expected congestion function, $E_{\mathbf{p}}[h[x_{k_i}, (x_{k_j})_{j \in N}]]$, as,

$$\begin{aligned} E_{\mathbf{p}} h[x_{k_i}, (x_{k_j})_{j \in N}] &= \left(E_{\mathbf{p}} h[m, (x_{k_j})_{j \in N}]\right)^2 Prob(a_i = m) + \left(E_{\mathbf{p}} h[e, (x_{k_j})_{j \in N}]\right)^2 Prob(a_i = e) \\ &= \left(E_{\mathbf{p}} h[m, (x_{k_j})_{j \in N}]\right)^2 x_{k_i} + \left(E_{\mathbf{p}} h[e, (x_{k_j})_{j \in N}]\right)^2 (1 - x_{k_i}). \end{aligned}$$

However, the expected proportion of players choosing action m, $E_{\mathbf{p}} h[m, (x_{k_j})_{j \in N}]$, is equal to the probability that a randomly chosen player has played action m. This probability is straightforwardly computed as,

$$
\begin{aligned}
E_{\mathbf{p}}h[m,(x_{k_j})_{j\in N}] &= Prob(\text{Randomly chosen player (r.c.p.) plays } m)\\
&= \sum_{l\in K} Prob(\text{r.c.p. plays } m|\text{r.c.p. has degree } l)Prob(\text{r.c.p. has degree } l])\\
&= \sum_{l\in K} x_l p_l
\end{aligned}
$$

Let $\overline{x}$ be the average proportion of the choices of all the network types,

$$\overline{x} = \sum_{l\in K} x_l p_l, \tag{6}$$

then the expected congestion function of a player of degree k, $E_{\mathbf{p}}[h[x_k,\mathbf{x}]]$, can be written as,

$$E_{\mathbf{p}}[h[x_k,\mathbf{x}]] = \frac{c}{2}\left[x_k\overline{x}^2 + (1-x_k)(1-\overline{x})^2\right], \tag{7}$$

where c is a parameter bigger than zero.

As the expected gross payoff function, the expected congestion cost is independent of the player's identity and depends on the player's own action, which is degree contingent, and on the expected choice of all the network players.

2.4. The expected net payoff function

Combining the two components of the value function, the expected net payoff function can be written as:

$$v_k[x_k,\mathbf{x}] = x_k\widetilde{x} + (1-x_k)(1-\widetilde{x}) - \frac{c}{2}\left[x_k\overline{x}^2 + (1-x_k)(1-\overline{x})^2\right]. \tag{8}$$

The structure of the expected net payoff function, where both gross payoffs and congestion are quadratic, can be found in other studies which analyze the effect of *local* externalities on players' decisions (see e.g. Galeotti and Vega-Redondo [14] or Ballester *et al.* [3]). Here, in contrast, we take into account both local and global externalities.

Then $v_k[x_k,\mathbf{x}]$ can be expressed as a quadratic function of x_k,

$$v_k[x_k,\mathbf{x}] = \alpha_{kk}x_k^2 + \Big(\sum_{l\in K\setminus k}\alpha_{kl}x_l + \beta_k\Big)x_k + \gamma_k(\{x_l\}_{l\neq k}) \tag{9}$$

where,

$$
\begin{aligned}
\alpha_{kk} &= 2\widetilde{p}_k - cp_k(\frac{1}{2}p_k + 1), && (10)\\
\alpha_{kl} &= 2\widetilde{p}_l - cp_l(p_k + 1), \text{ for all } l \neq k, && (11)\\
\beta_k &= cp_k - 1 + \frac{c}{2} - \widetilde{p}_k, \text{ and} && (12)\\
\gamma_k(\{x_l\}_{l\neq k}) &= (1 - \frac{c}{2}) - \sum_{l\neq k} x_l\widetilde{p}_l + \frac{c}{2}\big[\sum_{l\neq k} x_l p_l(2 - \sum_{l\neq k} x_l p_l)\big] && (13)
\end{aligned}
$$

Function $v_k[x_k, \mathbf{x}]$ depends on the weighted aggregation of the choices of all other player types in the network, i. e. $\sum_{l\neq k} \alpha_{kl}x_l$. Each weight α_{kl} measures the joint expected contribution of an l-degree player to the marginal payoffs of a k-degree player both as a neighbor and as a member of the whole network.

Notice that α_{kk}, the coefficient of the quadratic term, depending on the congestion cost can be either positive or negative and, therefore, the net payoff function can be either convex or concave. It is useful to know that from (10) to (13) the coefficients of type k's expected net payoff function verifies the following property,

$$
\alpha_{kk} + \beta_k + \frac{1}{2}\sum_{l\neq k} \alpha_{kl} = 0. \qquad (14)
$$

3. Existence of symmetric Bayesian Nash Equilibria

As it is well known a Bayesian Nash equilibrium always exists, possibly in mixed strategies, whenever functions $v_k[x_k, \mathbf{x}]$ are concave in x_k ($\alpha_{kk} < 0$), for all $k \in K$, and each strategy space is compact (as is in our case where $x_k \in [0, 1]$). However, by (10) above, concavity of all v_k's functions only results when c is sufficiently high.

When the congestion cost parameter c is small enough functions $v_k[x_k, \mathbf{x}]$ are convex in x_k ($\alpha_{kk} > 0$) in the interval $[0, 1]$, then best replies need not be continuous and have a discontinuity. This is so even for intermediate values of c, when some functions $v_k[x_k, \mathbf{x}]$ may still remain convex while others have already turned to be concave.[4] We need then to borrow results from supermodular games and monotone best responses (Vives [29]).

Function $v_k[x_k, \mathbf{x}]$ depends on x_k and on the choices of all other player types in the network, $\mathbf{x}_{-k}$. Theorem 4.2 in Vives (Vives [29]) states that a Bayesian Nash equilibrium exists if, for all $k \in K$, the set of strategies is a lattice compact, v_k is supermodular on $[0, 1]$ and/or has increasing differences in $(x_k, \mathbf{x}_{-k})$. Moreover, v_k is supermodular if and only if $\partial^2 v_k/\partial x_k \partial x_l \geq 0$ for all $l \neq k$. Given that $\partial^2 v_k/\partial x_k \partial x_l = \alpha_{kl}$, supermodularity is guaranteed if and only if $\alpha_{kl} \geq 0$ for all $k, l \in K$, $l \neq k$. On the other hand, v_k has increasing differences in $(x_k, \mathbf{x}_{-k})$ if $v_k[x_k, \mathbf{x}_{-k}] - v_k[x_k, \mathbf{x}'_{-k}]$ is increasing in x_k, for all $\mathbf{x}_{-k} \geq \mathbf{x}'_{-k}$ ($\mathbf{x}_{-k} \neq \mathbf{x}'_{-k}$).

[4] Since for $\alpha_{kk} = 0$, the value function is linear and we are only interested in quadratic value functions, then we disregard the linear case.

By (9) above,

$$v_k[x_k, \mathbf{x}_{-k}] - v_k[x_k, \mathbf{x'}_{-k}] = x_k \sum_{l \neq k} \alpha_{kl}(x_l - x'_l) + (\gamma_k(\mathbf{x}_{-k}) - \gamma_k(\mathbf{x'}_{-k})).$$

If $\mathbf{x}_{-k} \geq \mathbf{x'}_{-k}$, then v_k has increasing differences if and only if $\alpha_{kl} \geq 0$ for all $k, l \in K$, $l \neq k$, i.e., the same condition guarantees both supermodularity and increasing differences of v_k. Let $\Psi_k(\mathbf{x}_{-k})$ be the best response of type k to $\mathbf{x}_{-k}$, if v_k is either supermodular or has increasing differences and the strategy sets are lattices (sets [0,1]), then $\Psi_k(\mathbf{x}_{-k})$ is increasing in $\mathbf{x}_{-k}$, the composite best response is also increasing and (by Topkins' Theorem) a Bayesian Nash equilibrium exists (Vives [29]). Thus, *if each player considers each of the other players' action as a strategic complement* ($\alpha_{kl} \geq 0$ for all $l, k \in K$, $l \neq k$), *then Bayesian Nash equilibria will exis*t. Therefore we have two general sufficient conditions under which a Bayesian Nash equilibrium exists:

a) Either all the functions v_k are concave in x_k in the interval $[0, 1]$, or

b) Players are pairwise strategic complements ($\alpha_{kl} \geq 0$, for all $l \neq k$).

Let us define *homogeneous pure* strategies as the profiles where all types play the same pure action either m or e, i.e., the profiles $\mathbf{x_0} = (0, 0, \ldots, 0)$ and $\mathbf{x_1} = (1, 1, \ldots, 1)$; *heterogeneous pure* strategies as those profiles where all types play a pure action, not necessarily the same, i.e., those profiles $\mathbf{x}$ such that $x_k \in \{0, 1\}$ for all $k \in K$; *mixed* strategies as the profiles where all types play a mixed action ($x_k \in (0, 1)$ for all $k \in K$); and finally *hybrid* strategies as those profiles where some players choose a pure strategy and others a mixed strategy. Condition a) above is mainly concerned with the existence of mixed strategy Bayesian Nash equilibria but condition b) is a monotonicity condition which applies mainly to pure strategy and hybrid strategy Bayesian Nash equilibrium. In the following we relax condition b) and give conditions for the existence of the above different equilibrium profiles, which depend on both the concavity/convexity of the payoff functions and the aggregate strategic complementarity/substitution relationship between players' actions. Recall that we are only concerned with symmetric equilibria, i.e., the same type of players (players with the same k) will choose the same action.

3.1. Homogeneous pure strategy symmetric Bayesian Nash equilibria

Suppose firstly that all the v_k functions are convex in $[0, 1]$, then each $\Psi_k(\mathbf{x}_{-k}) \in \{0, 1\}$ and consider the two Bayesian Nash equilibria in homogeneous pure strategies, $\mathbf{x_0}$ and $\mathbf{x_1}$. In fact, from (9) it is easily seen that $v_k[\mathbf{x_1}] = v_k[\mathbf{x_0}]$ and that for $\mathbf{x}_{-k} = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2} \ldots \frac{1}{2})$, then both $x_k = 1$ and $x_k = 0$ belong to $\Psi_k(\mathbf{x}_{-k})$, so that best responses have at most a discontinuity. In this setting, we can notice by (9) and (14) that, for all k,

$$v_k[1, \mathbf{x}_{-k}] \geq v_k[0, \mathbf{x}_{-k}] \iff \sum_{l \neq k} (x_l - \frac{1}{2}) \alpha_{kl} \geq 0. \tag{15}$$

Thus, if $\mathbf{x}_{-k}$ is either a vector of 1's or of 0's, then whenever $\sum_{l \neq k} \alpha_{kl} \geq 0$, the best response of each player of type k, for all $k \in K$, is a non-decreasing function of the aggregate of the other players' strategies, being $x_k = 1$ as a best reply to a profile of 1's and a $x_k = 0$ to a profile of 0's. Hence, the equilibrium profiles are homogeneous sequences of either all 1's or all 0's. Thus, when functions v_k are all convex in $[0, 1]$, all we need to guarantee homogeneous pure strategy Bayesian Nash equilibria is that for any player, the other players' actions are strategic complements in the aggregate, i.e., $\sum_{l \neq k} \alpha_{kl} \geq 0$.

Similarly, suppose now that all the functions v_k were concave then,

$$\Psi_k(\mathbf{x}_{-k}) = \frac{\beta_k + \sum_{l \neq k} \alpha_{kl} x_l}{-2\alpha_{kk}}. \tag{16}$$

By concavity $\alpha_{kk} < 0$, homogeneous pure strategy equilibria would exist whenever $\beta_k \leq 0$ (corner solutions). In fact, by (14), this implies that $\sum_{l \neq k} \alpha_{kl} \geq 0$ and hence each player's best response function is non-decreasing in the aggregate of the other players' strategies.[5] These results can be extended to any mix of convex and concave payoff functions provided that for any player, the other players' actions are strategic complements in the aggregate, since in this case each player's best reply is a non-decreasing function of the aggregate of the other players' strategies. Thus,

Proposition 1. *The two homogeneous pure strategy symmetric BNE, $\mathbf{x_0}$ and $\mathbf{x_1}$, will exist if for any player, the other players' actions are* strategic complements in the aggregate*: for all $k \in K$, $\sum_{l \neq k} \alpha_{kl} \geq 0$. (If v_k is concave, then $\beta_k \leq 0$ will be sufficient to guarantee that condition).*

The above Proposition gives a weaker condition for the existence of homogeneous pure strategy BNE than supermodularity or increasing difference of functions v_k, where pairwise strategic complementarity is required.

Also notice that under the above conditions no *heterogeneous* pure strategy BNE will exist. However, these strategy equilibria may also exist under different conditions, as the following example shows.

Example 1. *Let $K = \{15, 16, 17\}$ with $p_k = 1/3$ for all $k \in K$. Then, $\widetilde{p}_{15} = 15/48$, $\widetilde{p}_{16} = 16/48$ and $\widetilde{p}_{17} = 17/48$.*

Here, functions v_k specify to (the terms not depending on x_k are not included):

[5] Alternatively, by (14) and (16) the condition for $\Psi_k(\mathbf{x}_{-k}) \geq 1$ is that $\sum_{l \neq k} \alpha_{kl}(x_l - 1) \geq \beta_k$ and then if $\mathbf{x}_{-k}$ is a vector of 1's all we need is that $\beta_k \leq 0$. Similarly as above, the condition for $\Psi_k(\mathbf{x}_{-k}) \leq 0$ is that $\sum_{l \neq k} \alpha_{kl} x_l \leq -\beta_k$, and then if $\mathbf{x}_{-k}$ is a vector of 0's, the result follows.

$$v_{15}[x_{15},(x_{16},x_{17})] = \alpha_{15,15}\, x_{15}^2 + [\alpha_{15,16}\, x_{16} + \alpha_{15,17}\, x_{17} + \beta_{15}]x_{15}$$
$$= (5/8 - (7/18)c)x_{15}^2 + \big[(2/3 - (4/9)c)x_{16} + (17/24 - (4/9)c)x_{17} + ((5/6)c - 21/16)\big]x_{15}$$

$$v_{16}[x_{16},(x_{15},x_{17})] = \alpha_{16,16}\, x_{16}^2 + [\alpha_{16,15}\, x_{15} + \alpha_{16,17}\, x_{17} + \beta_{16}]x_{16}$$
$$= (2/3 - (7/18)c)x_{16}^2 + \big[(5/8 - (4/9)c)x_{15} + (17/24 - (4/9)c)x_{17} + ((5/6)c - 4/3)\big]x_{16}$$

$$v_{17}[x_{17},(x_{15},x_{16})] = \alpha_{17,17}\, x_{17}^2 + [\alpha_{17,15}\, x_{15} + \alpha_{17,16}\, x_{16} + \beta_{17}]x_{17}$$
$$= (17/24 - (7/18)c)x_{17}^2 + \big[(5/8 - (4/9)c)x_{15} + (2/3 - (4/9)c)x_{16} + ((5/6)c - 65/48)\big]x_{17}$$

All v_k functions are convex and $\sum_{l\neq k}\alpha_{kl}{>}0$, whenever $c \leq 93/64 \approx 1.45$, for all $k \in \{15,16,17\}$. Then by Proposition 1, the unique symmetric BNE's are the two homogeneous pure strategy profiles $\mathbf{x}_0$ and $\mathbf{x}_1$.

However, for $99/64 \leq c < 45/28$, (aprox. $1.55 \leq c < 1.61$), all v_k functions are convex with $\alpha_{kl} < 0$, for all $l \neq k$, so that $\sum_{l\neq k}\alpha_{kl} < 0$. Since $\alpha_{16,17} \geq \alpha_{16,15}$ and $\alpha_{17,16} \geq \alpha_{17,15}$, the unique BNE's are the two heterogeneous pure strategy profiles $\mathbf{x} = (0,1,1)$ and $\mathbf{x}' = (1,0,0)$.

Notice that, interestingly enough, no symmetric equilibria will exist in the interval $93/64 < c < 99/64$. For instance, suppose that $c = 95/64$, then $\alpha_{kk} > 0$ for all $k \in \{15,16,17\}$, with $\sum_{l\neq k}\alpha_{15,l} > 0$, $\sum_{l\neq k}\alpha_{16,l} > 0$ and $\sum_{l\neq k}\alpha_{17,l} < 0$. It can be checked that homogeneous pure strategy equilibria do not exist. Neither do heterogeneous pure strategy equilibria. To see that, consider, as an example, the profile $\mathbf{x} = (0,1,1)$ as the proposed equilibrium. Type $k = 15$ will deviate and play $x_{15} = 1$ because $\sum_{l\neq k}\alpha_{15,l} > 0$ and then $v_{15}(1,1,1) > v_{15}(0,1,1)$. But for $(1,1,1)$ type $k = 17$ will deviate and play $x_{17} = 0$ because $\sum_{l\neq k}\alpha_{17,l} < 0$ and then $v_{17}(1,1,0) > v_{17}(1,1,1)$ and so on. Similar arguments apply for other heterogeneous profiles.

3.2. Heterogeneous pure strategy symmetric Bayesian Nash equilibria

Let K_0 be the set of types such that $x_k = 0$ and let K_1 be the set of types with $x_k = 1$, with $K_0 \cup K_1 = K$. First we analyze the conditions that characterize heterogeneous pure strategy equilibria when all functions v_k are convex. By (15), those conditions are $\sum_{l\in K_1\backslash k}\alpha_{kl} \geq \sum_{l\in K_0}\alpha_{kl}$, for all $k \in K_1$ and $\sum_{l\in K_1}\alpha_{kl} \leq \sum_{l\in K_0\backslash k}\alpha_{kl}$, for all $k \in K_0$. Notice that a sufficient condition for existence is that for all $k \in K_1$, the actions of the other players in K_1 are strategic complements in the aggregate, while those of the players in K_0 are strategic substitutes in the aggregate (i.e., $\sum_{l\in K_1\backslash k}\alpha_{kl} \geq 0 \geq \sum_{l\in K_0}\alpha_{kl}$); and equivalently for all $k \in K_0$, $\sum_{l\in K_0\backslash k}\alpha_{kl} \geq 0 \geq \sum_{l\in K_1}\alpha_{kl}$.

Similarly, assume now that all functions v_k are concave, then by the best reply function (see 16) and property (14), for all $k \in K_1$

$$\Psi_k(\mathbf{x}_{-k}) = \frac{\beta_k + \sum_{l\neq k}\alpha_{kl}x_l}{-2\alpha_{kk}} = \frac{\beta_k + \sum_{l\in K_1\backslash k}\alpha_{kl}}{2\beta_k + \sum_{l\neq k}\alpha_{kl}},$$

and $\Psi_k(\mathbf{x}_{-k}) \geq 1$ translates to condition $\sum_{l\in K_0} \alpha_{kl} \leq -\beta_k$. Equivalently, for all $k \in K_0$ since $\Psi_k(\mathbf{x}_{-k}) \leq 0$, then $\sum_{l\in K_1} \alpha_{kl} \leq -\beta_k$. If these conditions are satisfied an heterogeneous pure strategy equilibrium exists. Notice that by concavity and property (14) these two conditions imply that $\sum_{l\in K_0} \alpha_{kl} \leq \sum_{l\in K_1\setminus k} \alpha_{kl}$, for all $k \in K_1$ and $\sum_{l\in K_1} \alpha_{kl} \leq \sum_{l\in K_0\setminus k} \alpha_{kl}$, for all $k \in K_0$. These findings are summarized in next Proposition.

Proposition 2. *Let K_0 be the set of types such that $x_k = 0$ and let K_1 be the set of types with $x_k = 1$, with $K_0 \cup K_1 = K$, then heterogeneous pure strategy symmetric BNE exists whenever,*

1. Functions v_k are all convex, $\sum_{l\in K_1,\, l\neq k} \alpha_{kl} \geq \sum_{l\in K_0} \alpha_{kl}$ for all $k \in K_1$, and $\sum_{l\in K_1} \alpha_{kl} \leq \sum_{l\in K_0,\, l\neq k} \alpha_{kl}$ for all $k \in K_0$.

2. Functions v_k are all concave, $\sum_{l\in K_0} \alpha_{kl} \leq -\beta_k$ for all $k \in K_1$, and $\sum_{l\in K_1} \alpha_{kl} \leq -\beta_k$ for all $k \in K_0$.

We do not further proceed with the conditions for heterogeneous pure strategy symmetric equilibria when some functions are concave and some others convex.

3.3. Mixed strategy symmetric Bayesian Nash equilibria

As already mentioned, when functions $v_k[x_k, \mathbf{x}]$ are concave in x_k ($\alpha_{kk} < 0$), for all $k \in K$, and since each strategy space is compact a Bayesian Nash Equilibrium, possibly in mixed strategies, will always exit. Let us define the *uniformly mixed* strategy profile as the one where all players choose randomly between actions, giving each action the same probability of being chosen, i.e., the profile $\mathbf{x}^*$ such that $x_k^* = 1/2$ for all $k \in K$. If all payoff functions are concave, then the uniformly mixed strategy profile is a BNE. By (16),

$$\Psi_k(\mathbf{x}^*) = \frac{\beta_k + \frac{1}{2}\sum_{l\neq k} \alpha_{kl}}{-2\alpha_{kk}}$$

and by property (14), $-2\alpha_{kk} = 2(\beta_k + \frac{1}{2}\sum_{l\neq k} \alpha_{kl})$, then $\Psi_k(\mathbf{x}^*) = 1/2$.

Proposition 3. *The uniformly mixed strategy symmetric BNE exists whenever all payoff functions are concave.*

3.4. Hybrid symmetric Bayesian Nash equilibria

It remains to show the existence of *hybrid* symmetric BNE where some players choose a pure strategy and the others a mixed one. Given the difficulty of dealing with hybrid strategies, we restrict the analysis to those where players choosing a pure strategy select the same pure one. These equilibria can only take place when some v_k functions are convex and others are concave.

Let K_v be the set of types with concave payoff functions and let K_x be the one with convex payoff functions. Proceeding as above, the necessary and sufficient conditions in order a profile $\mathbf{x} = ((x_n)_{n\in K_v}, (x_k)_{k\in K_x})$ with $x_k = 1$ for all $k \in K_x$ and $x_n \in [\frac{1}{2}, 1]$ for all $n \in K_v$ with at least one $n_0 \in K_v$ such that $x_{n_0} < 1$ to be hybrid BNE are:

1. $\frac{1}{2}\sum_{l\in K_x\setminus k}\alpha_{kl}+\sum_{l\in K_v}(x_l-\frac{1}{2})\alpha_{kl}\geq 0$, for all $k\in K_x$, which guarantees that the players with convex function play action 1.

2. $\frac{1}{2}\sum_{l\in K_x}\alpha_{nl}+\sum_{l\in K_v\setminus n}(x_l-\frac{1}{2})\alpha_{nl}\geq 0$, for all $n\in K_v$, which guarantees that the players with concave function play action $x_n\in[\frac{1}{2},1]$.

3. There is at least an $n_0\in K_v$ such that $\beta_{n_0}+\sum_{l\in K_v\setminus n_0}(1-x_l)\alpha_{n_0l}>0$, which guarantees that the n_0-player type with concave function plays an action $x_{n_0}<1$.

The above necessary and sufficient conditions [6] translate to the set of sufficient conditions in terms of players strategic complementarity: all the players with convex functions satisfy that $\sum_{l\in K_x\setminus k}\alpha_{kl}\geq 0$ and $\alpha_{kn}\geq 0$ for all $k\in K_x$ and $n\in K_v$; and the players with concave function satisfy that $\sum_{l\in K_x}\alpha_{nl}\geq 0$, and $\alpha_{nl}\geq 0$ for all $n,l\in K_v$, $n\neq l$; and there is at least an $n_0\in K_v$ such that $\beta_{n_0}>0$. Strategic complementarity guarantee that all the players' best replies are non-decreasing and hence the composite best response is non-decreasing and (hybrid) equilibria exist.

A natural question to ask is whether profiles $\mathbf{x}=((x_n)_{n\in K_v},(x_k)_{k\in K_x})$ with $x_k=1$ for all $k\in K_x$, and $x_n\in[0,\frac{1}{2}]$ for all $n\in K_v$ at least one $n_0\in K_v$ such that $x_{n_0}>0$ can be supported as hybrid BNE.[7] To answer this question, suppose that all the payoff functions are concave but one, say v_k, this one being convex, and that the player with convex function consider as pairwise strategic substitutes the action of each other player (with concave function), i.e., $\alpha_{kn}\leq 0$, for all $n\in K_v$ and that those with concave functions also exhibit $\alpha_{nk}\leq 0$, for all $n\in K_v$. The idea of the proof relies on the fact that the system with concave functions has two solutions, parameterized by $x_k\in\{0,1\}$ (since the v_n are concave and the strategy space is compact), which are non-increasing in x_k because $\alpha_{nk}\leq 0$ for all n. Let $\mathbf{x}^*_{-\mathbf{k}}(1)=\{x^*_n(1)\}_{n\neq k}$ and $\mathbf{x}^*_{-\mathbf{k}}(0)=\{x^*_n(0)\}_{n\neq k}$, be such solutions. Since $\alpha_{kn}\leq 0$, for all $n\in K_v$, then the best reply $x_k\in\{0,1\}$ is also non-increasing in each solution and hence the composite best response is then non-decreasing and an equilibrium exists. More precisely, the necessary and sufficient conditions are:

1. $\sum_{l\in K_v}(x_l-\frac{1}{2})\alpha_{kl}\geq 0$, for k, which guarantees that the player with a convex function plays action 1.

2. $\frac{1}{2}\alpha_{nk}+\sum_{l\in K_v\setminus n}(x_l-\frac{1}{2})\alpha_{nl}\leq 0$ for all $n\in K_v$, which guarantees that the players with concave function play action $x_n\in[0,\frac{1}{2}]$.

3. There is at least an $n_0\in K_v$ such that $\beta_{n_0}+\sum_{l\in K_v\setminus n_0}\alpha_{n_0l}x_l+\alpha_{n_0k}>0$, which guarantees that at least a player with concave function plays an action bigger than 0.

Given that $\alpha_{kn}\leq 0$, for all $n\in K_v$, a sufficient condition for 1, i.e., that the player with the convex function plays action 1 is that $x_n\in[0,\frac{1}{2}]$, for all $n\in K_v$. Conversely, the players with concave functions will play $x_n\in[0,\frac{1}{2}]$, whenever $-\frac{1}{2}\alpha_{nk}\geq\sum_{l\in K_v\setminus n}(x_l-\frac{1}{2})\alpha_{nl}$. Notice that if $\alpha_{nk}\leq 0$, then this condition is trivially satisfied whenever $\alpha_{nl}\geq 0$ for all $l,n\in K_v$,

[6]Notice that if $\mathbf{x}$ is a BNE, then it is also the complementary profile $\mathbf{1}-\mathbf{x}$, and the same set of conditions guarantee both as BNE. Therefore, the conditions 1 to 3 are also necessary and sufficient in order a profile $\mathbf{x}=((x_n)_{n\in K_v},(x_k)_{k\in K_x})$ with $x_k=0$ for all $k\in K_x$ and $x_n\in[0,\frac{1}{2}]$ for all $n\in K_v$ with at least one $n_0\in K_v$ such that $x_{n_0}>0$ to be hybrid BNE.

[7]Or equivalently profiles $\mathbf{x}=((x_n)_{n\in K_v},(x_k)_{k\in K_x})$ with $x_k=0$ for all $k\in K_x$, and $x_n\in[\frac{1}{2},1]$ for all $n\in K_v$ at least one $n_0\in K_v$ such that $x_{n_0}<1$. See footnote 6.

$n \neq l$. Finally, to guarantee that at least one player with a concave function plays a mixed strategy (instead of the pure strategy opposite to the one played by the convex player) it is necessary that condition 3, which translates to require that $\beta_n \geq -\alpha_{nk} \geq 0$ for at least an $n \in K_v$.

Proposition 4. *Hybrid symmetric BNE (with homogeneous pure strategies) exist whenever some functions are convex and some others concave, and each player considers as strategic complements in the aggregate the convex function players' actions; each player considers as pairwise strategic complements the concave function players' actions; and $\beta_n > 0$ for at least a player with concave function. Let us denote as Type 1 the equilibria verifying these conditions.*

Consider that all functions are concave but one being convex, v_k and that each player with a concave function considers as pairwise strategic substitutes the convex players' actions and vice versa. Then, a Type 2 hybrid symmetric BNE exists when either i) the players with concave value functions exhibit pairwise strategic complementarity, i.e., each $\alpha_{nl} \geq 0$ for all $l, n \in K_v$, $n \neq l$ or ii) the players with concave value functions exhibit pairwise strategic substitution, i.e., $\alpha_{nl} \leq 0$ for all $l, n \in K_v$, $n \neq l$ and the parametrized vector of solutions $\mathbf{x}^*_{-\mathbf{k}}(1)$ and $\mathbf{x}^*_{-\mathbf{k}}(0)$ are sufficiently close to 1/2.

3.5. Symmetric equilibrium may fail to exist when set K is finite.

The above Propositions give us conditions which are sufficient to guarantee the existence of a different kind of symmetric BNE. However, there still remains the question of whether symmetric equilibrium may fail to exist. The original game has a countable infinite number of players, whose pure strategy sets are compacts and their payoff functions are continuous. Therefore, as Salonen states (Salonen [27]) a Nash equilibrium in mixed strategies must exist. The existence failure may come from the fact that in symmetric equilibria the players' strategy profiles are indexed by the players' degree, so that the original game (with a countable infinite number of players) translates to a new game where the number of players is set K. Thus, when K is finite it would be as if the underlying game we are dealing with had a finite set of players. In this situation, the existence of symmetric Nash equilibria need not be guaranteed when some best response functions are increasing while some other ones decreasing, thus possibly missing the equilibrium. Notice also that all of our results are under the assumption of independence between neighbors's degree and individual degree, and we are not sure about what affiliation between players' degrees may add to the analysis.

The following example with three type of players illustrates the non-existence problem as well as the above Propositions.

Example 2. *Let g be a network where individuals are unformly distributed in degrees 2,3 and 4. Then $K = \{2, 3, 4\}$ and $p_k = 1/3$, for all $k \in K$.*

Here, functions v_k specify to (the terms not depending on x_k are not included):

$$v_2[x_2,(x_3,x_4)] = (4/9-(7/18)c)x_2^2 + [(2/3-(4/9)c)x_3 + (8/9-(4/9)c)x_4 + ((5/6)c - 11/9)]x_2$$
$$v_3[x_3,(x_2,x_4)] = (2/3-(7/18)c)x_3^2 + [(4/9-(4/9)c)x_2 + (8/9-(4/9)c)x_4 + ((5/6)c - 4/3)]x_3$$
$$v_4[x_4,(x_2,x_3)] = (8/9-(7/18)c)x_4^2 + [(4/9-(4/9)c)x_2 + (2/3-(4/9)c)x_3 + ((5/6)c - 13/9)]x_4$$

Notice that v_2 is convex for $c \leq 8/7 \approx 1.14$, v_3 is convex for $c \leq 12/7 \approx 1.714$ and v_4 is convex for $c \leq 16/7 \approx 2.29$.[8]

For $c < 1.14$, we find two symmetric BNE in homogeneous pure strategies, where conditions of Proposition 1 are fulfilled, i.e., all the v_k functions are convex with $\sum_{l\neq k}\alpha_{kl}>0$, for $k \in K$, i.e., actions of the players are strategic complements in the aggregate.

For $c > 1.14$ function v_2 turns to be concave while v_3 and v_4 still remain convex. Nevertheless, in the interval $1.14 < c \leq 1.25$, $\sum_{l\neq k}\alpha_{kl} \geq 0$, for $k \in K$ (with $\beta_2 < 0$), the conditions of Proposition 1 still are satisfied and we find that the unique symmetric equilibria are the two equilibria in homogeneous pure strategies.

As above, in the interval $1.25 < c < 1.71$, both functions v_3 and v_4 are still convex and v_2 concave but *symmetric BNE fail to exist*. In particular, in the interval $1.25 < c < 1.50$, where $\sum_{j\neq 2}\alpha_{2j} > 0$, $\sum_{j\neq 3}\alpha_{3j} > 0$, but $\sum_{j\neq 4}\alpha_{4j} < 0$. The reason is the failure of type $k = 4$ to consider the other players' actions as strategic complements in the aggregate; similarly, in the interval $1.50 < c < 1.71$, $\sum_{j\neq 2}\alpha_{2j} > 0$ but $\sum_{j\neq 3}\alpha_{3j} < 0$ and $\sum_{j\neq 4}\alpha_{4j} < 0$, and thus players' actions are neither strategic complement nor strategic substitutes in the aggregate. The same argument applies to the interval $1.71 < c < 2$, where both v_2 and v_3 are now concave while v_4 remains convex, with $\sum_{j\neq 2}\alpha_{2j} > 0$ but $\sum_{j\neq 3}\alpha_{3j} < 0$ and $\sum_{j\neq 4}\alpha_{4j} < 0$.

For $c \geq 2$, the equilibrium is restored. In the interval $2 \leq c < 2.29$, v_4 is convex and v_2 and v_3 are concave with $\alpha_{kl} < 0$ for all $k \neq l$, so that actions are all pairwise strategic substitutes. For instance, for $c = 2.2$, the two hybrid BNE are $\{x_2^* = 0.51, x_3^* = 0.60, x_4^* = 0\}$ and $\{x_2^* = 0.49, x_3^* = 0.40, x_4^* = 1\}$ (see Type 2 hybrid symmetric BNE, ii) in Proposition 4.)[9]

Finally, for $c > 2.29$, all the v_k functions, are concave. The unique symmetric BNE is the uniformly mixed strategy profile $\{x_2^* = 0.5, x_3^* = 0.5, x_4^* = 0.5\}$ (see Proposition 3).

The equilibrium configuration is displayed in Figure 1. In this example, the existence failure of symmetric Bayesian Nash equilibrium, for some values of the congestion parameter, comes from the fact that K is finite and for this range of c, players' actions are neither strategic substitutes nor strategic complements.[10]

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[8] Also notice that for $c < 1$, $\alpha_{kl} > 0$, for $k, l = 2, 3, 4$ and $k \neq l$. For $1 \leq c \leq 1.25$, $\sum_{l\neq 4}\alpha_{4l} \geq 0$ (although $\alpha_{42} < 0$), for $1 \leq c \leq 1.5$, $\sum_{l\neq 3}\alpha_{3l} \geq 0$ (although $\alpha_{32} < 0$) and for $1 \leq c \leq 1.75$, $\sum_{l\neq 2}\alpha_{2l} \geq 0$ (although $\alpha_{23} < 0$ for $c \geq 1.5$).

[9] Notice that $\alpha_{24} = -0.0888$, $\alpha_{23} = -0.311$, $\alpha_{22} = -0.411$ and $\beta_2 = 0.613$;$\alpha_{34} = -0.0888$, $\alpha_{32} = -0.533$, $\alpha_{33} = -0.1888$ and $\beta_3 = 0.499$. Hence, $\beta_n > -\alpha_{n4} > 0$, (or $\alpha_{nn} > -\frac{1}{2}\alpha_{n4}$) for $n \in \{2, 3\}$ and $x_2^*(0) = 0.51, x_3^*(0) = 0.60$, and $x_2^*(1) = 0.49, x_3^*(1) = 0.40$ are sufficiently close to $1/2$ such that $\alpha_{nl}(x_l(1) - \frac{1}{2}) + \frac{1}{2}\alpha_{n4} < 0$, and $\alpha_{nl}(x_l(0) - \frac{1}{2}) - \frac{1}{2}\alpha_{n4} > 0$ for $n \in \{2, 3\}$, $l \neq n$.

[10] If we removed the restriction of equilibrium symmetry, then we could search for the Bayesian

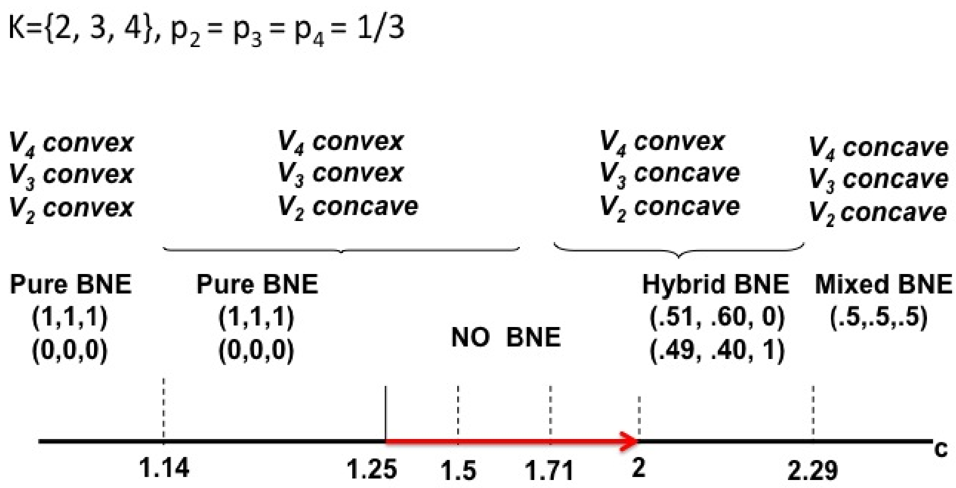

Fig. 1. Equilibrium configuration for Example 2 as a function of the congestion parameter.

4. Network characterization of Bayesian Nash Equilibria

As we have seen BNE profiles depend on both the degree distribution $\mathbf{p}$ of the network and the congestion parameter c. Intuition suggests that if congestion is high enough, the unique BNE profile is the one in which players' choice of actions are as heterogeneous as possible. Only for low congestion will the players choose the same action.

However, intuition has to be polished since *the network global topology plays an important role in the equilibrium characterization.* To appreciate this notice that the degree distribution defines two important network features such as *hub* and *peripheral players.* Although each individual's value function depends on both the average action profile followed by all the individuals of the network and the average profile of their neighbors, their relative weight will depend on the individual's number of connections. Thus, the network average action profile is particularly important for peripherals because, by definition, their number of neighbors is very small and therefore their choices will mostly be driven by the network global topology. On the contrary, the hubs choices will mainly depend on their neighbors average action profile, i.e. on the network's local properties. Therefore both local and global properties determine the equilibrium choices. The proportion of hubs and peripherals depends on the weight of the tails of the degree distribution and as a consequence so does the equilibrium characterization.

The next results characterize the BNE profiles of Section 3: *mixed strategy profiles, homogeneous pure strategy profiles* and *hybrid equilibrium profiles*, in terms of the network

Nash equilibria of the original game, where each player chooses individually their action. In this framework, the existence of equilibrium is restored: it can be checked that the profile with all players playing the same pure strategy is a Bayesian Nash equilibrium whenever $c \leq 2$. Given a player, say i, if all the other players choose the same pure action, say m –i.e., $x_j = 1$ for all $j \neq i$– then the player i's gross payoff is x_i (because the average action of their neighbors is 1), and the player i's cost is $\frac{c}{2}x_i$ (because the average action of the network is also 1). Then, player i's net payoff is $x_i - \frac{c}{2}x_i$, and their best response is $x_1 = 1$ if $c \leq 2$, and $x_i = 0$ otherwise.

topology. First, Proposition 3 can be expressed as,

Proposition 5. *Let g be a network with a degree distribution of $\mathbf{p} = \{p_k\}_{k \in K}$. The unique mixed strategy BNE is the uniformly mixed strategy profile. Moreover, the uniformly mixed strategy will be a BNE if and only if the network relative degree $\frac{k}{d}$ is bounded from above: $\frac{k}{d} < \frac{c}{4}(2 + p_k)$ for all $k \in K$.*

The first statement in Proposition 5 says that if $\mathbf{x}$ is a mixed strategy BNE, i.e. $x_k \in (0, 1)$ for all $k \in K$, then it cannot be otherwise unless $x_k = 1/2$. The second one gives a necessary and sufficient condition, concavity of the v_k functions, for all $k \in K$, in order that the uniformly mixed strategy profile is a BNE. Given a congestion function parameter, concavity imposes an upper bound on the maximum relative degree of the considered network. This means that the right tail of the degree distribution tells us whether a uniformly mixed strategy BNE exists. Therefore, *uniformly mixed BNE profiles are very difficult to achieve in networks with players with high relative degree (hubs) unless the congestion cost parameter is very high. This is so even when there is only one such a player. In fact, when the maximum degree k is not bounded, then mixed strategy equilibrium will not exist.* (Some examples are in the next Section.)

Let us give some intuition. Hubs always have an incentive to coordinate their actions and select the same pure strategy. Suppose that all players with different degrees from k choose the uniformly mixed strategy. If k is big enough, then $k\widetilde{p}_k$ will be also big enough, which means that the k-degree players will have many k-degree neighbors. Thus, if these players chose a pure strategy, say action e, their increase in the gross payoff would be high and would compensate for the increase in their congestion cost. More precisely, if all players chose the uniformly mixed strategy their utility function, according to (8), would be $\frac{1}{2} - \frac{c}{8}$. Now, if the k-degree players changed their strategy and all of them choose action e, their value function would be $\frac{1}{2}(1 + \widetilde{p}_k) - \frac{c}{8}(1 + p_k)^2$. Notice that now both the gross payoff and the congestion cost are higher than before. This deviation is not profitable as long as $\frac{1}{2} - \frac{c}{8} > \frac{1}{2}(1 + \widetilde{p}_k) - \frac{c}{8}(1 + p_k)^2$, which implies that $\frac{c}{4}p_k + \frac{c}{8}p_k^2 > \frac{1}{2}\widetilde{p}_k$. Recalling that $\widetilde{p}_k = kp_k/d$, this inequality is equivalent to the condition of the above Proposition.

An alternative interpretation would arise if the inequality of Proposition 5 was re-written in terms of a threshold on the congestion cost. Thus, the uniformly mixed strategy BNE will exist if and only if $c > max_{k \in K} \frac{4k}{d(2+p_k)}$. Hence, uniformly mixed profiles will appear whenever there are no hubs in the network or congestion is very high.

Next, we characterize the existence conditions of homogeneous pure strategy BNE's. The following Proposition translates Proposition 1 to conditions on the network degree distribution.

Proposition 6. *Let g be a network with a degree distribution of $\mathbf{p} = \{p_k\}_{k \in K}$. An homogeneous pure strategy will be a BNE if one of the following conditions is satisfied, either*

$$\frac{c}{4}(2 + p_k) \leq \frac{k}{d} \leq \frac{1}{p_k}(1 - \frac{c}{2} + \frac{c}{2}p_k^2) \tag{17}$$

for all value of k, or (17) is satisfied for some values of k and

$$\frac{1}{p_k}(cp_k - 1 + \frac{c}{2}) \leq \frac{k}{d} < \frac{c}{4}(2 + p_k) \tag{18}$$

for the other values of k.

The left hand side inequality in condition (17) implies that v_k is convex, and the the right hand side inequality implies that $\sum_{l \neq k} \alpha_{kl} \geq 0$. With respect to condition (18), the right hand side inequality implies that v_k is concave and the left hand side that $\beta_k \leq 0$ (and $\sum_{l \neq k} \alpha_{kl} \geq 0$). Let us interpret the above results in terms of hubs, peripherals and the congestion cost parameter.

The convexity of the v_k functions is trivially satisfied for low values of c and the right hand side inequality of (17) puts a constraint on the degree probability distribution: namely, this inequality is satisfied whenever $p_k \leq \frac{1-c/2}{k/d}$ for all $k > d$. Therefore, given a low congestion cost parameter, the existence of a homogeneous pure strategy BNE imposes an upper bound on the weight of the right-tail of the degree distribution $\mathbf{p}$, i.e. in the accumulative probability of hubs. Notice that now there is not an upper bound in the maximum relative degree (as was necessary for the uniformly mixed strategy BNE, see Proposition 5), but instead *hubs have to be quite unlikely.* The reason is the following: let us assume that all players, except the k-degree ones, choose action m. If the expected number of k- degree neighbors of a k-degree players, $k\widetilde{p}_k$, is high enough, then it will be very likely that these players will be linked to other k-degree players. If k-degree players choose the other pure action, e, then their reduction on the gross payoff will be low and may be offset by the reduction in their congestion cost. To avoid this deviation, $k\widetilde{p}_k$ must be low enough. Therefore, when k is (relatively) high, as it is for hubs, p_k must be low enough.

When c takes intermediate values, relatively high values of k/d have to satisfy condition (17) and relatively low ones condition (18). The left hand side inequality of (18) can be expressed as $p_k \leq \frac{1-c/2}{c-k/d}$, a bound on the left-tail of the degree distribution $\mathbf{p}$. Thus, the proportion of peripherals has to be low in order an homogeneous pure strategy BNE to exist. The reason is the following: peripherals only suffer congestion costs and receive hardly any gross payoffs. Thus, if all players choose action m, then peripherals will have incentives to switch to action e because their gross payoff will not change but their congestion cost will be drastically reduced. Therefore, for moderate values of the congestion cost, condition (17) implies that hubs have to be unlikely and condition (18) says that peripherals have to be so as well. Under high values of the congestion parameter homogeneous pure strategy profiles cannot be BNE. If c is high enough, then neither the left hand side inequality in (17) nor the left hand side inequality in (18) will be satisfied by any value of the relative degree.

To sum up, homogeneous pure strategy BNE will exist if hubs are quite unlikely whenever the congestion parameter is low enough; if the congestion parameter takes intermediate values, then the existence of homogeneous pure strategy equilibrium profiles will be ensured as long as both hubs and peripherals remain unlikely; finally, there will not be an homogeneous pure strategy BNE if the congestion parameter is high. Notice that since the choice of peripherals is mainly driven by the global network topology, while that of hubs is determined instead by the local network topology, both local and global externalities play a role in the existence of homogeneous pure strategy equilibrium choices.

Finally, notice that the conditions for existence of hybrid equilibria given in Proposition 4 are more complex and demanding over the (pairwise) relationship between players. This new requirements put additional bounds on the network maximum relative degree. The translation of Proposition 4 to the network topology is as follows.

Proposition 7. *Let g be a network with a degree distribution of $\mathbf{p} = \{p_k\}_{k\in K}$. A hybrid symmetric Bayesian Nash equilibrium will exist of either type 1 or type 2 if:*

Type 1: For all type k with a convex function and n with a concave function, i.e. for all $k\in K_k$ and for all $n \in K_v$

$$\begin{aligned} \frac{c}{4}(2+p_k) &\leq \frac{k}{d} && (19)\\ \max_{l\in K\backslash n}\{\frac{c}{2}(1+p_l)\} &\leq \frac{n}{d} \leq \frac{c}{4}(2+p_n) && (20) \end{aligned}$$

with at least one $n_0 \in K_v$ such that $\frac{n_0}{d} < \frac{1}{p_{n_0}}\left[\frac{c}{2}(2p_{n_0}+1)-1\right]$; and

$$\begin{aligned} d_{x,k} &\geq \frac{c}{2}d(1+p_k) && (21)\\ d_x &\geq \frac{c}{2}d\max_{l\in K_v}\{(1+p_l)\} && (22) \end{aligned}$$

where d_x is the average degree of all the players with convex value function and $d_{x,k}$ is the average degree of all the players with convex value function except those with degree k, i.e., $d_x = \sum_{l\in K_x} lp_l/\sum_{l\in K_x} p_l$ and $d_x = \sum_{l\in K_x\backslash k} lp_l/\sum_{l\in K_x\backslash k} p_l$; or

Type 2: For the unique type k with a convex function, and for all $n \in K_v$,

$$\begin{aligned} \frac{c}{4}(2+p_k) \leq \frac{k}{d} &\leq \frac{c}{2}(p_n+1) && (23)\\ \frac{n}{d} &\leq \min\{\frac{c}{2}(p_k+1), \frac{c}{4}(p_n+2)\} && (24) \end{aligned}$$

with at least one $n_0 \in K_v$ playing a mixed strategy.

As in Proposition 6, the inequality in condition (19) and the right hand side inequality in condition (20) imply that v_k is convex and v_n is concave, respectively. Moreover, the left hand side inequality in (20) implies that all players' actions are pairwise strategic complements with those of players with concave functions, i.e., $\alpha_{ln} \geq 0$ for all $l \in K$ and all $n \in K_v$. Finally, conditions (21) and (22) imply that each player considers as strategic complements in the average the convex function players' actions, i.e., $\sum_{l\in K_x\backslash k} \alpha_{kl} \geq 0$ and $\sum_{l\in K_x} \alpha_{nl} \geq 0$.

With respect to the second set of conditions: the left hand side inequality in (23) and the right hand side inequality in (24) imply that v_k is convex and v_n is concave, respectively; the right hand side inequalities in both (23) and (24) imply that the convex function players' actions are pairwise strategic substitutes with those of the concave function players, and vice versa.

The first set of conditions does not impose either a constraint on the existence of both hubs and peripherals or a constraint on their likelihood. However, the left hand side inequality in (20) is only verified when the number of the concave function players is bounded from above. Moreover, the average degree of the players with convex value functions is bounded from below, and this bound increases proportionally with the network average degree. This translates to requiring that hubs must be likely enough.

The second set of conditions implies that there are no hubs in the network. Even more, the players with the highest degree should be very unlikely with respect to the peripherals' likelihood.

In type 1 equilibria both hubs and peripherals should be very likely, so that the formers choose the same pure action to benefit from their interaction within them. On the contrary, peripherals, who are very frequent, prefer a mixed action as a way to reduce their congestion cost. In type 2 equilibria instead, where there are no hubs (set K must be finite), the best peripherals' strategy is to play a mixed strategy in order to reduce congestion cost as much as possible (see Example 2).

5. Equilibrium analysis in some random networks

The equilibrium analysis of two common degree probability distributions of random networks illustrates the above results. Empirical analysis of social networks and theoretical models on the dynamic of network formation conclude that the most common random networks are the Poisson network and the Scale-free network, where the former can be translated to a Poisson degree distribution and the latter to a Scale-free degree distribution, also referred as the power-law degree distribution (Newman [22], Albert and Barabasi [1] and Jackson [18]). Both distributions have *fat tails*, but that of a Scale-free distribution is *fatter* than the one with a Poisson distribution. That is, the proportion of nodes with both low and large degrees are higher than could be expected if the links were formed completely independently as occurs in Poisson random networks (see the left graph in Figure 2). In our terminology, peripherals and hubs are unlikely in Poisson random networks in comparison with Scale-free random networks, where both are more frequent. As set K is countably infinite under both distributions, Bayesian Nash equilibria always exist. In addition, since the maximum degree is not bounded, uniformly mixed Bayesian equilibrium will not exit, by proposition 5. All equilibria can be found are the pure strategy and hybrid Bayesian Nash equilibria.

The relationship between the congestion parameter and the different kind of BNE under Poisson networks exhibits a complex behavior, which depends on the network average degree. The right graph in Figure 2 displays the equilibrium strategies for a Poisson degree distribution with average degree equal to 3 and for three congestion cost parameters, namely 1, 3 and 10 (solid, dashed and dotted lines respectively). For the lowest congestion parameter, $c = 1$, all type of players choose the same pure strategy ($x_k = 1$ for all $k \in K$); when the congestion parameter is higher, $c = 3$, the players with a degree lower than 5 play a mixed strategy, since these player value functions have turned to be concave; and the remaining players, with still convex value functions, choose the same pure strategy ($x_1 = 0.56, x_2 = 0.47, x_3 = 0.45, x_4 = 0.63, x_k = 1$ for $k \geq 5$). For the highest congestion parameter, $c = 10$, more player value functions turn to be concave and thus players with a degree lower than 13 play a mixed strategy and players with a bigger degree play the pure

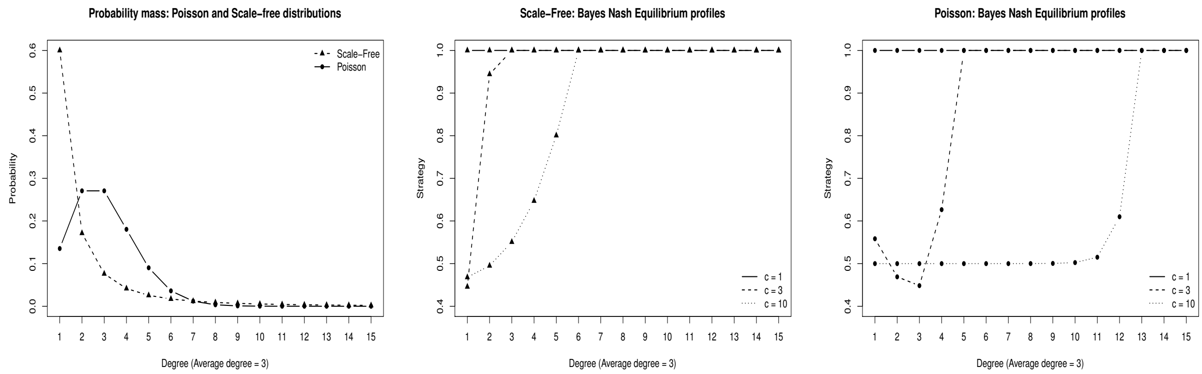

Fig. 2. Left: Probability mass of a Poisson distribution (solid line) and a Scale Free distribution (dotted line) both with an average degree of 3. Center: Equilibrium strategies under a Scale-Free distribution as a function of the players' degree, for congestion parameters equal to 1, 3 and 10. Right: Equilibrium strategies under a Poisson distribution as a function of the players' degree, for congestion parameters equal to 1, 3 and 10.

strategy $x_k = 1$, for $k \geq 13$).[11] Recall that since k is unbounded there are no mixed equilibria (see Proposition 5).

In general, let us consider a Poisson random network with an average degree equal to $d = \lambda + 1$, i.e., $p_k = e^{-\lambda}\lambda^{k-1}/(k-1)!$ Under Proposition 6 homogeneous symmetric BNE conditions exist when either all value functions are convex (see condition (17)) or both convex and concave value functions coexist (see condition (18)). In a Poisson random network condition (17) translates to requiring $c \leq \frac{4}{(\lambda+1)(2+e^{-\lambda})}$, which implies that all the v_k functions are convex and players' actions are strategic complements in the aggregate; similarly condition (18) translates to requiring that $c \leq \frac{4k_0}{(\lambda+1)(2+p_{k_0})}$ where k_0 approaches the average degree, $\lambda + 1$, from below.[12] In term hybrid strategies are the only possible BNE whenever the congestion parameter is big enough or v_k is concave for some k greater than the average degree, $\lambda + 1$. Notice that hybrid symmetric BNE is of type 1 since the there are always a countable infinite number of convex functions. As above mentioned no mixed BNE exist as k is unbounded.

In Table 1 we relate the network average connectivity with homogeneous symmetric BNE. More precisely, we show the congestion parameter's range for homogeneous symmetric BNE both with all value functions being convex and with some value functions being concave. The c-range is provided for a sample of network average degrees. For example, if the Poisson network average degree is 10, and $c < 0.20$ then all value functions are convex and homogeneous symmetric BNE exist; this result is extended to $.20 < c < 1.76$ but the players with lower degree have a concave

[11]Nevertheless, if the network average degree is high enough, new equilibrium could be observed where both peripherals and hubs play the same pure strategy and the players with a degree sufficiently close to the network average degree play a mixed strategy.

[12]More preciely, k_0 is the maximum degree such that $\frac{4(k_0-1)}{(\lambda+1)(2+p_{k_0-1})} \leq \min_{k \leq k_0 - 1} 2\frac{1+\widetilde{p}_k}{1+2p_k}$.

value function. Notice that the higher the average degree the lower the c-range for convex value functions and the higher the c-range for concave ones. In the limit, the upper bound of c that guarantees the existence of homogeneous symmetric BNE is 2.

Scale-free random networks show a simpler relationship between the congestion cost parameter and the different symmetric BNE than Poisson random networks do. The graph in the center of Figure 2 displays the symmetric equilibrium strategies of a Scale-Free random network with average degree equal to 3 and for the same congestion parameters than before. For a congestion parameter equal to 1, all type of players choose the same pure strategy; when the congestion parameter is equal to 3, the players with degree 1 and 2 play the mixed strategy $x_1 = 0.45$, $x_2 = 0.94$ and the remaining players choose the pure strategy $x_k = 1,\ k \geq 3$; for $c = 10$, the players with a degree lower than 6 play the a mixed strategy, and those with a bigger degree play the pure strategy $x_k = 1,\ k \geq 6$.

Let us consider a Scale-free random network: $p_k = (\gamma - 1)B(k, \gamma)$, for all $k \geq 1$, $2 < \gamma \leq 3$ and where $B(a, b)$ is the Legendre beta-function.[13] Then the average degree is $d = (\gamma - 1)/(\gamma - 2)$. For homogeneous symmetric BNE condition (17) in Proposition 6 translates to requiring that $c \leq \frac{4\gamma(\gamma-2)}{(\gamma-1)(3\gamma-1)}$, while condition (18) requires $c \leq \frac{4(\gamma-1)}{3\gamma-2}$. Whenever v_k is concave for k greater than half the average degree or the congestion cost parameter is big enough type 1 hybrid strategies are the only possible symmetric BNE.

As we know by Proposition 7, type 1 hybrid BNE exists even when both peripherals and hubs are very likely, as under Poisson and Scale-free distributions. However in both distributions, contrary to Proposition 7, the number of concave function players is not bounded from above (this is so because this Proposition only states sufficient but not necessary conditions).

By Proposition 6, since Scale-free tales are fatter than those of Poisson we should expect both a smaller range of congestion costs and a more reduced number of concave value functions in the former than in the latter under which homogeneous symmetric BNE exist. Table 1 confirms these results. In the Scale-free distribution we observe the same pattern as in Poisson distribution: the higher the average degree the lower the c-range for which all functions are convex and the higher the c-range for which concave and convex functions co-exit. However, independently of the network average degree, the c-range in Poisson random networks is always higher than the one in Scale-free networks.

6. Some comparative statics for the two-type player case

Propositions 5, 6 and 7 relate the different kind of Bayesian Nash equilibria with the proportion of hubs in the network and their probability distribution, for given congestion costs. As we have seen it is very complex to undertake a general comparative static analysis. However, for a population with two-types of players something more definite can be said. We display here the graphical results of two examples which clearly illustrate the above Propositions. Namely, how the equilibrium configuration changes under the presence of hubs and under the increase in their probability.

[13] In other words $B(a, b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$., where Γ is the Gamma function

Table 1. c-range for homogeneous BNE under Poisson and Scale-free distributions.

Network Average degree, d	Poisson		Scale free	
	All convex	Concave/Convex	All convex	Concave/Convex
3	$0-0.6244$	$0.6244-1.5317$	$0-0.5128$	$0.5128-1.0910$
5	$0-0.3964$	$0.3964-1.6629$	$0-0.3130$	$0.3130-1.0526$
8	$0-0.2499$	$0.2499-1.7362$	$0-0.1974$	$0.1974-1.0323$
10	$0-0.2000$	$0.2000-1.7639$	$0-0.1583$	$0.1583-1.0256$
20	$0-0.1000$	$0.1000-1.8319$	$0-0.0796$	$0.0796-1.0127$
50	$0-0.0400$	$0.0400-1.8921$	$0-0.0319$	$0.0319-1.0050$
100	$0-0.0200$	$0.0200-1.9229$	$0-0.0160$	$0.0160-1.0025$
∞	$0-0.0000$	$0.0000-2.0000$	$0-0.0000$	$0.0000-1.0000$

Consider first how equilibria changes under the presence of hubs and peripheral, keeping the degree probability distribution constant. Suppose that $K = \{3, 4\}$ with $p_3 = p_4 = 0.5$ as opposed to $K = \{3, 40\}$ with $p_3 = p_{40} = 0.5$. Thus, the first example illustrates the situation where there are neither peripherals nor hubs in the population, while in the second example the average degree is equal to 21.5, a half of the population (the 3-degree players) consists of peripherals and the other half (the 40-degree players) is composed by hubs. Figure 3 displays the symmetric BNE as a function of the congestion parameter. The left hand side of Figure 3 corresponds to the first example while the right hand side displays the second one. Graphs are scaled to enable comparison of the range of existence of the different symmetric BNE.[14]

When $K = \{3, 4\}$ -neither peripherals no hubs-, the two homogeneous pure strategy BNE (either $(0, 0)$ or $(1, 1)$) exist up to a congestion cost of 1.14. Then, there is a congestion parameter interval –from $c = 1.14$ to $c = 1.52$– for which there is no equilibrium. (It can be checked that the best response functions of the underline two type player game do not cross each other at any point). For values of the parameter from 1.52 to 1.83 we find two hybrid BNE with the 4-degree player choosing a pure strategy (either 1 or 0) and the 3-degree player the corresponding mixed one. Finally, for high congestion cost, the uniformly mixed BNE is the unique equilibrium.

[14]The reader can check that when $K = \{3, 4\}$, the players' expected payoff functions are

$$\begin{aligned} v_3(x_3, x_4) &= (c - 10/7)x_3 + (6/7 - 5c/8)x_3^2 + (8/7 - 3c/4)x_3x_4 + \gamma_3 \\ v_4(x_4, x_3) &= (c - 11/7)x_4 + (8/7 - 5c/8)x_4^2 + (6/7 - 3c/4)x_4x_3 + \gamma_4 \end{aligned}$$

while these functions are when $K = \{3, 40\}$,

$$\begin{aligned} v_3(x_3, x_{40}) &= (c - 46/43)x_3 + (6/43 - 5c/8)x_3^2 + (80/43 - 3c/4)x_3x_{40} + \gamma_3 \\ v_{40}(x_{40}, x_3) &= (c - 83/43)x_{40} + (80/43 - 5c/8)x_{40}^2 + (6/43 - 3c/4)x_{40}x_3 + \gamma_{40} \end{aligned}$$

$K = \{3, 4\}$, $p_3 = p_4 = 0.5$ $\qquad$ $K = \{3, 40\}$, $p_3 = p_{40} = 0.5$

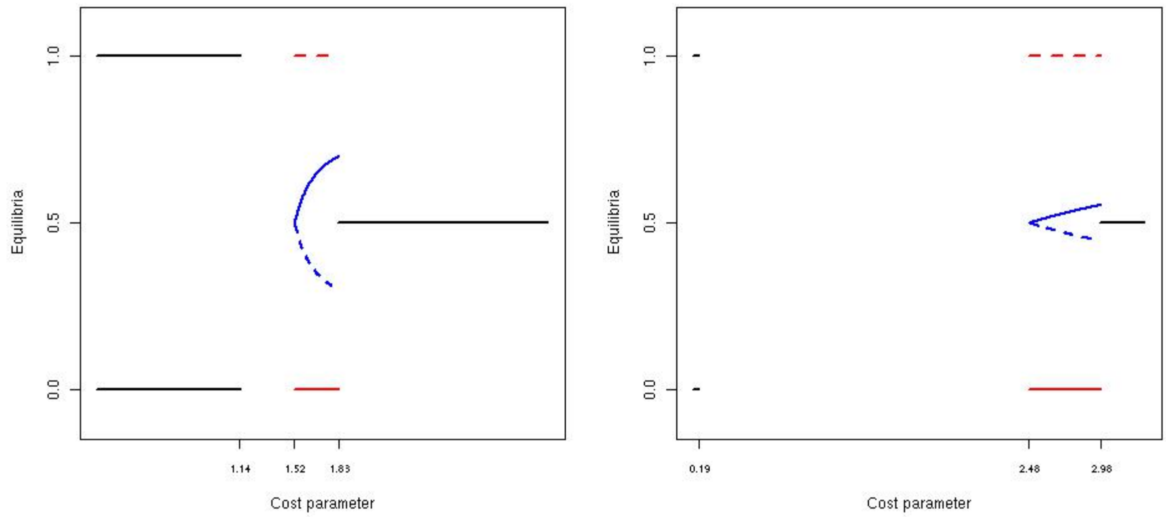

Fig. 3. Symmetric BNE of two-type players as a function of the congestion parameter, under degree configurations with and without hubs. The solid lines indicate that both players play the same strategy, either homogeneous pure profiles or uniformly mixed profiles; dashed lines together with solid lines mean that each player plays a different strategy (hybrid equilibria). In this case, the dashed lines are one of the possible hybrid equilibrium and the solid lines are the other one.

When $K = \{3,\ 40\}$, both peripherals and hubs are present. By Propositions 5, 6 and 7 this implies with respect to the previous case: i) a decrease in the range of existence of homogeneous pure strategy equilibria (the upper bound of c is now 0.19); ii) a decrease in the existence of uniformly mixed BNE (the lower bound of c is now 2.98); iii) an increment in the range of non-existence, being now from $c = 0.19$ to $c = 2.48$; iv) and an approach to 1/2 of the mixed strategy of the 3-degree players in the hybrid equilibria. Hence, the presence of both peripherals and hubs reduces the range of congestion cost where either pure or mixed equilibrium profiles exist.

However, as above mentioned, the degree probability distribution also plays a key role in both the equilibrium configuration and its existence. To illustrate this point, consider again the above two-type player networks with either $K = \{3, 4\}$ or $K = \{3, 40\}$ and the three probability distributions, $p_3 = 0.25$, $p_3 = 0.50$ and $p_3 = 0.75$. Figure 4 displays the change in the BNE's configurations when both the probability distribution and the ratio between degrees change. In the top of Figure 4 the two type of players have similar degree, $K = \{3, 4\}$ and in the bottom there are peripherals and hubs, $K = \{3, 40\}$. In addition, in the two graphs on the left hand side of the Figure $p_3 = 0.25$, the middle hand side $p_3 = 0.50$ and finally, on the right hand side $p_3 = 0.75$. In other words, the probability of hubs decreases as we move to the right.[15]

If $p_3 = 0.5$ and $p_4 = 0.5$, then see footnote 14.

If $p_3 = 0.75$ and $p_4 = 0.25$, then

$$\begin{aligned} v_3(x_3, x_4) &= (5c/4 - 22/13)x_3 + (18/13 - 33c/32)x_3^2 + (8/13 - 7c/16)x_3x_4 + \gamma_3 \\ v_4(x_4, x_3) &= (3c/4 - 17/13)x_4 + (8/13 - 9c/32)x_4^2 + (18/13 - 15c/16)x_4x_3 + \gamma_4. \end{aligned}$$

For $K = \{3, 40\}$, if $p_3 = 0.25$ and $p_4 = 0.75$, the players' expected payoffs are

$$\begin{aligned} v_3(x_3, x_{40}) &= (3c/4 - 42/41)x_3 + (2/41 - 9c/32)x_3^2 + (80/41 - 15c/16)x_3x_{40} + \gamma_3 \\ v_{40}(x_{40}, x_3) &= (5c/4 - 81/41)x_{40} + (80/41 - 33c/32)x_{40}^2 + (2/41 - 7c/16)x_{40}x_3 + \gamma_{40}. \end{aligned}$$

If $p_3 = 0.5$ and $p_4 = 0.5$, then see footnote 14.

If $p_3 = 0.75$ and $p_4 = 0.25$, then

$$\begin{aligned} v_3(x_3, x_{40}) &= (5c/4 - 58/49)x_3 + (18/49 - 33c/32)x_3^2 + (80/49 - 7c/16)x_3x_{40} + \gamma_3 \\ v_{40}(x_{40}, x_3) &= (3c/4 - 89/49)x_{40} + (80/49 - 9c/32)x_{40}^2 + (18/49 - 15c/16)x_{40}x_3 + \gamma_{40}. \end{aligned}$$

Inspection of Figure 4 reveals some facts related with the degree probability distribution. When the probability of hubs decreases three general facts are observed. First, the range of congestion costs where there exists homogeneous pure symmetric BNE increases (see Proposition 6). In the three top graphs, this range moves from c in $(0, : 0.91)$ to c in (0, 1.34); in the bottom graphs the change is less relevant, being now from c in (0, 0.11) to c in (0, 0.36). Second, the range of congestion costs where uniformly mixed symmetric BNE exist decreases (see Proposition 5). Again, in the three top graphs, the lower bound of the congestion cost for which there are uniformly mixed BNE moves from 1.55 to 2.19, and in the bottom graphs it moves from 1.89 to 5.80. And third, the range of congestion costs for which there are hybrid symmetric BNE increases: for instance, consider the bottom graphs, on the left hand side there is not any hybrid BNE, while in the middle hand side, hybrid BNE exist in the interval $c \in (2.48, 2.98)$, and in the right hand side they exist for $c \in (3.73, 5.80)$.

However, since K is finite there is always an interval of the congestion parameter where symmetric BNE fail to exist. The length of this interval depends on both the degree probability distribution and the ratio between degrees, showing however a complex behavior. To see this observe that if players have similar degree and the probability of the maximum degree players decreases, then the length of this interval will also decrease. This is the case

[15]For $K = \{3, 4\}$, if $p_3 = 0.25$ and $p_4 = 0.75$, the players' expected payoffs are

$$\begin{aligned} v_3(x_3, x_4) &= (3c/4 - 6/5)x_3 + (2/5 - 9c/32)x_3^2 + (8/5 - 15c/16)x_3x_4 + \gamma_3 \\ v_4(x_4, x_3) &= (5c/4 - 9/5)x_4 + (8/5 - 33c/32)x_4^2 + (2/5 - 7c/16)x_4x_3 + \gamma_4. \end{aligned}$$

displayed in the three top graphs, where the length of the interval of c precluding the equilibrium existence is reduced from a measure of 0.64 to one of 0.08. While, if players' degree are far apart, then this length will increase as the probability of the maximum degree players decreases. In the three bottom graphs, the length of the interval of c where the BNEs fail to exist increases from a measure of 1.78 to one of 3.37.

We can extend the above example results to a general two type player case. Denote by k and l the two type player degrees with $k > l$ and let $p = p_l$. We show the following results that explain and generalize the examples.[16] Firstly, as k increases, v_k is convex for more values of c while v_l is convex for fewer values of c. Secondly, if $k > 2l$, then as p increases (the probability of peripherals increases) both functions v_k and v_l will be convex for more values of c. However, when $k < 2l$ (no hubs), v_l will be convex for fewer values of c. Thirdly, as k increases the values of c for which type k considers the action of type l as a strategic complement will be reduced. The opposite results takes place for type l with respect to type k. Combining these last results with the first one above it can be said that, *with p fixed, as k increases the range of values of c for which homogeneous symmetric BNE exist will decrease.* This is so because v_k is convex for more values of c, but v_l is convex for fewer values of c, the range of values of c for which type k considers the action of type l as a strategic complement is smaller, but the range of values of c for which type l considers the action of type k as a strategic complement is bigger. These opposite effects translate to a lack of players' coordination and thus to a bigger interval of non-existence results, as displayed in Figure 3. Finally, for any two type degrees k and l such that $k > \frac{l(1+2p)}{2p}$ (the case with hubs) as the probability of peripheral increases the range of c for which type k (type l) considers the action of type l (k) as a strategic complement will increase as well. Therefore, *with k and l fixed and k sufficiently high (hubs) as p increases the range of values of c for which homogeneous symmetric BNE exist will increase* (see the bottom graphs in Figure 4). When $k < \frac{l(1+2p)}{2p}$ (no hubs), as p increases type l will turn to consider the type k's action as a strategic substitute for lower values of c, thus resulting in an increase of hybrid symmetric BNE for more values of c as p increases, as shown in the top of Figure 4. Hence, *with k and l fixed and k sufficiently low (no hubs) as p increases the range of values of c for which hybrid symmetric BNE exist will increase.*

To illustrate the complex behavior between the degree probability distribution and the ratio between degrees and the non-existence of symmetric BNE we display Figure 5. This figure shows for specific values of the ratio l/k (from 0.1 to 0.9) the length of the congestion parameter interval where symmetric BNE fail to exist (exe OY) as a function of the p (exe OX). For example, when $l/k = 3/30$ and $p = .20$ corresponds to .20 in exe OX and line .1, so that the length c is around 1.7. Notice that if the ratio l/k is small enough (there are hubs), then as p increases the non-existence range of c will increases as well. On the contrary, for higher values of l/k (no hubs), the relationship is more complex, with a non-existence range decreasing in p at the beginning and increasing latter on. The opposite behavior is observed for intermediate values of the ratio l/k.

[16]We do not include the cumbersome calculation which are a tedious verification of the existence propositions. These calculations are available from the authors upon request.

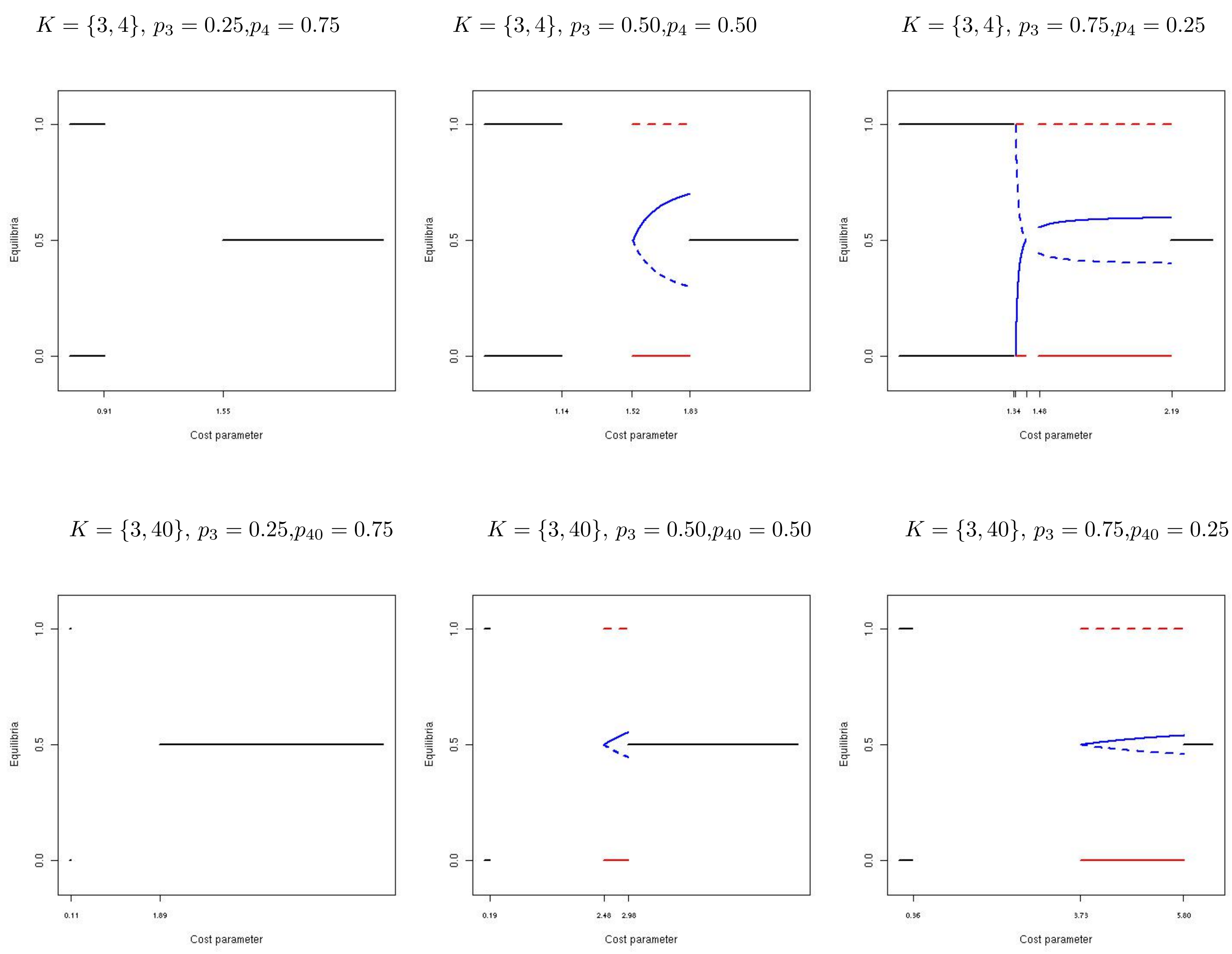

Fig. 4. Symmetric BNE of two-type players as a function of the congestion parameter, under different degree configurations and probability distributions.

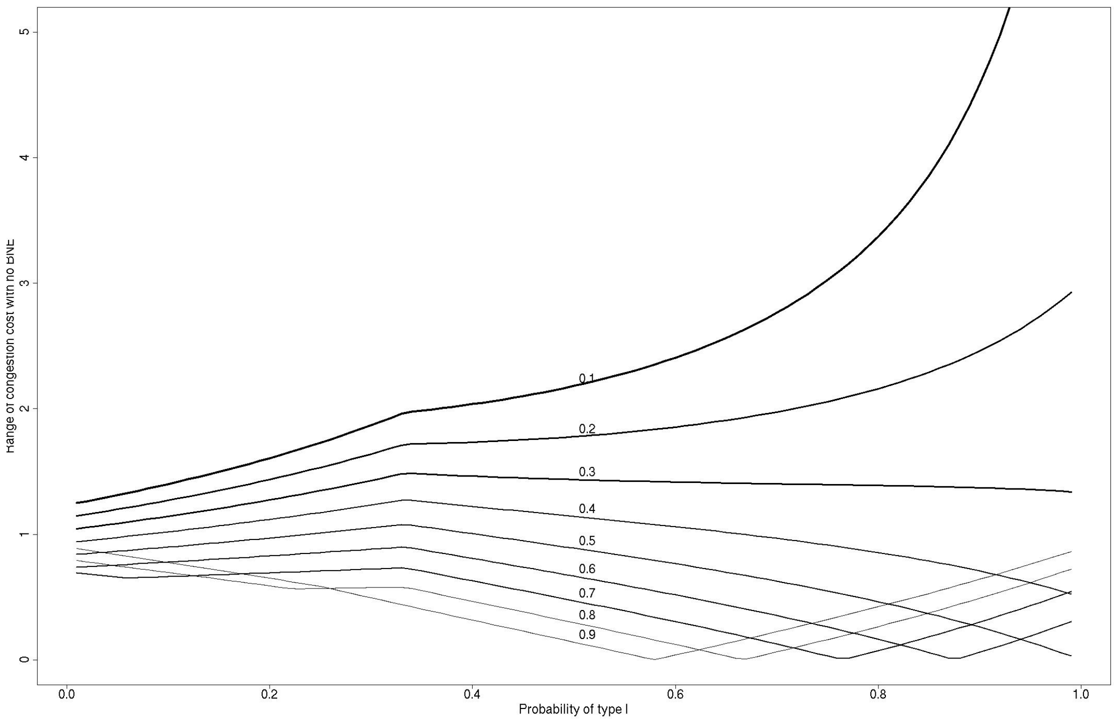

Fig. 5. The measure of the range of c for the two type player non-existence problem. In the OX exe the probability of the lower degree type p, and in the OY exes the range of congestion costs with non-existence of symmetric BNE. Each line corresponds to a different ratio between the lower degree type and that of the higher degree type l/k, from 0.1 (top) to 0.9 (bottom).

7. Conclusions

This paper analyzes the impact of local and global interaction on individuals' choices. Players are located in a network and interact with each other with perfect knowledge of their own neighborhood and probabilistic knowledge of the complete network topology.

Individuals simultaneously choose their actions from a finite set, which imposes an externality on their neighbors as well as an externality on the complete network, and then obtain a utility. Namely, players obtain utility from sharing their choices with their neighbors (positive local externality) but suffer disutility from sharing that choice with all the members of the network (negative global externality). A variety of economic and social phenomena exhibit these features such as the adoption of cost-reducing innovations, clusters of firms, the choice of time-schedules, etc. The optimal (Bayesian Nash) decision taken by each individual depends on three factors: the spread of their connections in the network (their degree), their knowledge about the network's topology, and the exact nature of the externalities which impact on their utility.

Our main contribution is to show that both local and global network properties play an important role in equilibrium choices. This is so because the network topology defines two important features such as *hubs* (highly connected nodes) and *peripherals* (poorly connected nodes). Although each individual's value function will depend on both the average action profile followed by the network and the average action profile of their neighbors, their relative weight will depend on the individual's number of connections. Thus, the network average action profile is particularly important for peripherals because, by definition, their number of neighbors is very small and therefore their choice will mostly be driven by the network global topology. On the contrary, the hubs or highly connected players' action choices will mainly depend on the average profile of their neighbors' actions, i.e. on the network local properties. Therefore, our symmetric Bayesian Nash Equilibrium is expressed in terms of the ratio between hubs and peripherals which, in turn, comes from both the asymmetry of the degree probability distribution (its skewness) and the weight of its tails.

Finally and as a by-product of our analysis, we have also found that if the set of degrees is finite, a non-existence problem may appear when dealing with symmetric equilibrium. The measure of the congestion parameter for which the equilibrium does not exist depends on both the hub existence and their probability, but this dependence is not monotonic and rather complex.

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