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Multiproduct trading with a common agent under complete information: Existence and characterization of Nash equilibrium☆

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Abstract

This paper focuses on oligopolistic markets in which indivisible goods are sold by multiproduct firms to a continuum of homogeneous buyers, with measure normalized to one, who have preferences over bundles of products. Our analysis contributes to the literature on private, delegated agency games with complete information, extending the insights by Chiesa and Denicolò (2009) to multiproduct markets with indivisibilities and where the agent's preferences need not be monotone. By analyzing a kind of extended contract schedules -mixed bundling prices- that discriminate on exclusivity, the paper shows that efficient equilibria always exist in such settings. There may also exist inefficient equilibria in which the agent chooses a suboptimal bundle and no principal has a profitable deviation inducing the agent to buy the surplus-maximizing bundle because of a coordination problem among the principals. Inefficient equilibria can be ruled out by either assuming that *all* firms are pricing unsold bundles at the same profit margin as the bundle sold at equilibrium, or imposing the solution concept of subgame perfect *strong equilibrium*, which requires the absence of profitable deviations by any subset of principals and the agent. We also provide a characterization of the equilibrium strategies. More specific results about the structure of equilibrium prices and payoffs for common agency outcomes are offered when the social surplus function is monotone and either submodular or supermodular.

Keywords: Multiproduct Price Competition, Delegated Agency Games, Mixed Bundling Prices, Subgame Perfect Nash Equilibrium, Strong Equilibrium. *JEL Classification:* C72, D21, D41, D43, L13.

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1. Introduction

This paper focuses on oligopolistic markets in which indivisible goods are sold by multiproduct firms to a continuum of homogeneous buyers, with measure normalized to one, who have preferences over bundles of products. In these settings linear pricing does not guarantee the existence of efficient Nash-equilibrium outcomes and, even worse, sometimes equilibrium (either efficient or inefficient) fails to exist. We wish to investigate whether a kind of non-additive prices, *mixed bundling prices*, restores equilibrium existence and efficiency. Mixed bundling refers to the practice of offering a consumer the option of buying goods separately or else packages of them (at a discount over the single good prices).

Our analysis contributes to the literature of delegated agency games extending their insights to multiproduct markets with indivisibilities and where the agent‘s preferences need not be monotone. In our model, the principals are multi-product oligopolists offering a menu of prices for the different bundles of their own indivisible products and the agent is the consumer. We tackle the general question of whether the mixed-bundling equilibrium contracts offered by multiple principals to an agent will be efficient and how the social surplus will be split among them. Obviously a single firm can achieve efficiency and extract all surplus in this setting when it can price non-additively. However, when prices charged by one firm impose some contractual externalities on other firms it is far from clear why equilibrium should be expected to be efficient.

When several firms sell (non-homogeneous goods) to the same consumer using some price scheme as a price discrimination strategy, the price schedule which arises can also be modeled as an equilibrium to a common agency game. It is natural to allow the consumer the option of purchasing exclusively from one firm, or from a set of firms, and so common agency is no longer intrinsic to the game but a choice variable that is delegated to the agent. Furthermore, in many economics contexts, several principals contract with a common agent and each of those principals is directly affected by the action selected by the common agent regarding other principals. In such a context, there are contractual externalities among principals. For instance, Martimort and Stole (2003), analyze the equilibrium set of a simple common agency (or "bidding") game: the set of outcomes which results from non-cooperative behavior among the principals when they compete through non-linear prices. They analyze two variations of the common agency game: the intrinsic common agency game and the delegated common agency game. Our model directly applies to the analysis of delegated agency games with complete information. More particularly, we extend the abstract model of trading of Chiesa and Denicolò (2009) to cover multiproduct markets with indivisibilities and an agent with preferences over bundles of products. Like theirs we also analyze unrestricted offers. In this setting we show the equilibrium existence and efficiency and characterize the set of equilibrium payoffs and strategies.

We also contribute to the understanding of the interaction between competition and non-additive pricing schemes. Because most of the theoretical work in multi-principal contract games has restricted attention in large part to intrinsic settings, it remains unclear how competition affects the character of non-additive prices among oligopolists. Clear exceptions, among others, are Armstrong and Vickers (2001) and Rochet and Stole (2002), who analyze nonlinear pricing applications assuming exclusive purchasing in which the consumer must buy from only one firm in equilibrium, and competition can only matter through the outside option; Ivaldi and Martimort (1994), who allow for purchasing from multiple vendors in equilibrium but restrict preferences such that full coverage arises in equilibrium; and Martimort and Stole (2009), who study how competition in non-linear pricing between two principals (sellers) affects market participation by a privately-informed agent (consumer). In Martimort and Stole (2009), each firm produces a good (which can be a substitute or complement of the rival) and the consumer's demand function for the two goods is symmetric and linear in prices (with the consumer's private information appearing only in the demand intercept). In contrast, in our complete information model each firm produces a set of goods, the consumer has a value function over bundles of goods and both multiple vendors (common agency) and exclusivity are possible equilibrium outcomes. In our analysis, the strategy space of principals are the set of firms' strategies in the complete information game of firms and the consumer. Thus, we consider extended contract schedules instead of only the equilibrium offers. There are at least two reasons for focusing on the complete information case. First, the literature on common agency games deals mainly with principals selling one or two substitute goods under continuity assumptions so that the general existence theorems (when available) for such games cannot be easily extended to cover models with indivisible goods and where the agent's preferences need not be monotone, even under complete information.[1] Second, ruling out informational considerations allow us to isolate the effect on the outcome of the principals' competition game from the effects stemming from private information. In particular, this permits us, in calculating the equilibrium payoff of the agent, to know the magnitude of the rent she obtains as the result of competition among principals.

Finally, the paper is also related to the literature of bundling and mixed bundling literature. The literature on multi-product pricing focuses on the use of bundling to extract a surplus from heterogeneous buyers or to price-discriminate (Adams and Yellen (1976); Schmalensee (1984); MacAfee *et al.* (1989)) or to apply monopoly leverage across markets (Whinston (1990); Choi (1996, 2004); Carbajo *et al.* (1990)) or to deter entry into the market (Whinston (1990); Nalebuff (1999, 2004)). In all these models, demands are assumed to be continuous, there is private information and consumers are heterogeneous. In contrast, in our setting with indivisible good, mixed bundling is profitable when there is a representative consumer and no opportunity to apply leverage across markets, even under complete information.[2]

[1]For existence results in common agency games under incomplete information see, Carmona and Fajardo (2009), who show that general menus games satisfying enough continuity properties have subgame perfect equilibrium. See also Page and Monteiro (2003).

[2]Mixed bundling also takes place with substitute goods, in contrast to the well-studied cases of bundling with independent or complementary goods. There are few general results for bundles of more than two goods. McAdams (1997) found that the existing analytical machinery for analyzing mixed bundling could not be easily generalized to even three goods, because of the interactions among sub-bundles. In general, price-setting for mixed bundling of many goods is an NP-complete problem requiring sellers to determine a number of prices and quantities that grows exponentially as the size of the bundle increases. Therefore,

To the best of our knowledge, our paper is the first one dealing with multiproduct oligopolistic competition when goods are indivisible, of a very general nature and the agent's preferences over bundles of goods need not be monotonic. For instance, in many common situations, agents have complementary preferences for objects in the marketplace. Consider an agent trying to construct a computer system by purchasing components. Among other things, the agent needs to buy a CPU, a keyboard and a monitor, and may have a choice over several models for each component. The agent's valuations of a package depends on the components in any particular combination, involving products of either only one firm or several firms. This example is a general instance of allocation problems characterized by heterogeneous, discrete resources and complementarities in agents' preferences. These kinds of models are probably close to many circumstances in real world markets, but they are also more difficult to analyze. With indivisibilities, it is well-known that many familiar properties of the profit functions may fail to ensure the existence of Nash-Bertrand prices. The use of marginal calculus is precluded, and the applications of fixed point theorems based on continuity properties, while still possible in some cases, is certainly not straightforward.

Furthermore, equilibrium efficiency may require additional restrictions in our model. For instance, in a typical (not necessarily efficient) delegated common agency (subgame perfect) Nash-equilibrium, the prices of the equilibrium bundles are well-specified, but any vector of sufficiently high prices for the individual components (out-of-equilibrium prices) would support any given equilibrium outcome. In this way, if a principal reduces the price of any of its unsold individual components, then the equilibrium consumption set is not upset since rivals' prices for their individual components are also too high to induce the agent to choose an alternative bundle to that of the (not necessarily efficient) equilibrium. This leads to well-known inefficiencies, since efficiency may imply some coordination among firms (as stressed by Martimort (2007)), specially if the efficient consumption is a multi-firm bundle (common agency). Thus, we need a refinement of the (subgame perfect) Nash-equilibrium concept which would require the candidate equilibrium to remain so even if all firms reduced the prices of their unsold bundles to some degree, thus restoring coordination and efficiency.

The Strong Equilibrium (Aumann (1959)) is the equilibrium concept capturing the above idea of stability against joint deviations that are mutually profitable for *any* subset of players. The extension to subgame perfection of such an equilibrium concept allows us to refine the equilibrium correspondence by selecting its efficient equilibria. In fact, we show that the set of subgame perfect strong equilibria of our game is the set of its efficient subgame perfect Nash-equilibria. Therefore inefficient equilibria are be ruled out imposing the solution concept of strong equilibrium which requires the absence of profitable deviations by any subset of principals and the agent.

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a general analysis is still lacking and, what is worse, it is not even known whether a Nash equilibrium may exist in such a general setting. In a related setup, Liao and Urbano (2002) and Liao and Tauman (2002) consider a duopoly and assume that each firm produces two complementary goods which are substitutes for the two corresponding goods produced by the other firm. Liao and Tauman (2002) find that mixed bundling strategies play a key role in stabilizing the market, although efficient and inefficient equilibria may exist. If the use of mixed bundling is not allowed, then Liao and Urbano (2002) show that subgame perfect linear pricing equilibria may fail to exist.

The question of equilibrium efficiency was already addressed by Bernheim and Whinston (1986b), in the context of delegated common agency and where each principal can contract on the whole array of actions of the agent. Under complete information the so-called *truthful equilibrium* implements the outcome which maximizes the aggregated payoff of the grand coalition made of the principals and the agent. The rationale for truthful menus is that they are *coalition-proof*, i.e., immune to deviations by subsets of principals which are themselves immune to deviations by sub-coalitions, etc. Coalition-proof equilibrium payoffs can be implemented with truthful schedules in environments with quasi-linear utility functions. Unfortunately, in more complex settings, where the social value function function need not be either concave or convex and what is more it may even fail to be continuous, the refinement of coalition-proof equilibrium may be too demanding to be satisfied. We impose instead the strong equilibrium refinement, which takes into account *any* deviation by any set of firms and the player.

The paper offers a positive existence result: with mixed-bundling contracts and subgame perfect strong equilibria, *an efficient equilibrium outcome always exists* in our agency game, no matter whether the efficient bundle is either exclusive dealing or common agency (involving either several principals or all of them). The paper extends the traditional wisdom of the delegated common agency literature to settings with multi-product principals producing indivisible goods and an agent with preferences non necessarily monotone over bundles of goods. By analyzing mixed-bundling contracts that discriminate on exclusivity and extend the space of contract schedules beyond equilibrium offers the paper provides 1) a proof of the existence of efficient subgame perfect Nash-equilibria and hence that of strong equilibria in delegated agency games, 2) a characterization of such equilibria, 3) a characterization of the set of the principals' equilibrium rents by some projection of the core of such agency games. Namely, the principals' equilibrium payoffs need not be unique and are characterized by the polytope formed by their most preferred points in the core of the delegated agency game, where the maximum payoff of a firm is bounded from above by its marginal contribution. 4) Finally, we show that in delegated agency equilibrium outcomes, i.e., when the agent buys to a subset of principals, each principal's set of contracts of *minimum cardinality* (Chiesa and Denicolò (2009)), will contain at least three offers to support the equilibrium outcome.

Specific results about the structure of equilibrium prices and payoffs for common agency outcomes are offered when the social surplus function is monotone and either submodular or supermodular. In the former case, principals are substitutes and their equilibrium rents are equal to their social marginal contributions with an agent's positive rent, thus reflecting market competition. In the latter case, the agent's rent is zero and then the core of the value function is always priced by the subgame perfect Nash-equilibrium rents. These results allow us conclude that the lack of monotonicity of the agent's preferences over bundles of goods and hence the inherited lack of monotonity of the social value function is the reason for our results on the minimum cardinality of each principal's set of contracts and on the principals' rents of the general model.

The paper is organized as follows. The model is presented in Section 2. The characterization of efficient subgame perfect Nash-equilibrium in terms of all possible deviations and hence that of subgame perfect strong equilibrium are provided in Section 3. The existence and efficiency of such equilibria are proven in Section 4. Section 5 offers specific results for common agency equilibrium outcomes when the social surplus function is monotone and either submodular or supermodular. Concluding remarks close the paper.

2. The model

Consider a set of the principals and a continuum of potential homogeneous buyers, with measure normalized to one (the agent). In our model the principals are n firms and each of them produces a finite set of heterogeneous goods. Moreover each firm's products can be different from or identical to those of any other firm. Let $N = \{1, 2, ..., n\}$ be the set of firms. Let Ω_i be firm i's finite set of products and $\Omega = \Omega_1 \times \ldots \times \Omega_n$ be the cartesian product of the sets of all firms. Let $c_i(w_i)$ be the (constant) unit cost of production of firm i for good $w_i \in \Omega_i$, where costs are additive, i.e. $c_i(T_i) = \sum_{w \in T_i} c_i(w), T_i \subseteq \Omega_i$. Trade is modeled as a complete information first-price auction in which principals simultaneously submit a menu of contracts and the agent then chooses the set of products she will purchase from each principal.

A consumption set is a vector of subsets $\mathbf{S} = (S_1, \ldots, S_n) \subseteq \Omega$, where $S_i \in 2^{\Omega_i}$ represents firm i selling set S_i in $\mathbf{S}$ (which can be the empty set if the agent does not buy anything from firm i). Let $c(\mathbf{S}) = \sum_{i \in N} c_i(S_i)$ be the cost of the consumption set $\mathbf{S}$ (where $c_i(\emptyset) = 0$).

A firm is said to be active in a given consumption set if some of its products is consumed, and non-active otherwise. Let $F(\mathbf{S})$ be the set of active firms in $\mathbf{S}$, i.e. $F(\mathbf{S}) = \{i \in N | S_i \neq \emptyset\}$.

The agent purchases either one or zero units of each one of the products and is characterized by her value function over any subset $\mathbf{S} \subseteq \Omega$, $v(\mathbf{S})$, which represents her total willingness to pay for consumption set $\mathbf{S}$, with $v(\emptyset, \ldots, \emptyset) = 0$.[3] Initially we do not impose any assumption on the value function. It is assumed that the agent has no endowment of goods but she has enough money to buy any bundle of products $\mathbf{S} \subseteq \Omega$.

To keep the model as general as possible, we set no restriction on feasible contracts by firms.[4] Namely, in our setting we do not require principals to submit only one offer for each product of the equilibrium outcome but allow them to submit offers for each subset of their product sets. Thus, since the sets traded by the agent with other principals are not contractible, each generic firm i sets prices (offers contracts) for its products and can also offer subsets of them as bundles at a special price. Notice that this implies that we do

[3] The value function can be derived by standard primitives. Suppose that the agent's utility function is quasilinear in money, that is, it is given by $u(x_1, ..., x_n, m) = f(x_1, ..., x_n) + m$, where m is the monetary numerare and $(x_1, ..., x_n)$ is a consumption bundle. Then, $f(x_1, ..., x_n)$ measures the monetary value of the bundle $(x_1, ..., x_n)$. Let $S \subseteq N$ be a consumption set and let e^S be the corresponding quantities consumed, namely, $e^S_k = 1$ if $k \in S$ and $e^S_k = 0$ if $k \notin S$. The value function v is defined as $v(S) = f(e^S)$.

[4] This is in contracts with settings with a perfectly divisible homogeneous product where is analytically convenient allowing principals not to submit any offer for certain output levels, as in multi-unit-autions or markets for electricity, where principals offer a finite number of contracts.

not consider singleton contracts (direct mechanisms) in the delegated agency game. Such mechanisms do not allow for any offer to remain unchosen in equilibrium; in other words, out-of-equilibrium messages are possible, to use the language of mechanism design.

A strategy of firm $i \in N$ is a 2^{Ω_i}-tuple specifying the price of each $w \in \Omega_i$ as well as the price of each of any other subset of Ω_i. Let $p_i(T_i)$ be the price of $T_i \subseteq \Omega_i$. To avoid irrational off-equilibrium behavior, we restrict $p_i(T_i)$ to satisfy $p_i(T_i) \geq c_i(T_i)$ and set $p_i(\emptyset) = 0$. Let $\mathcal{P}_i = \mathbb{R}_+^{2^{\Omega_i}}$ be the set of firm i's strategies , i.e., the set of functions $p_i : 2^{\Omega_i} \longrightarrow \mathbb{R}_+$ such that $p_i \geq c_i$. If $p_i(T_i) = \sum_{w \in T_i} p_i(w)$, then prices are linear and bundle T_i is not offered at a special price. If $p_i(T_i) < \sum_{w \in T_i} p_i(w)$, then subset T_i is offered as a bundle at a lower price. In this case, we say that firm i follows a *mixed bundling strategy*.

After each firm i has chosen a price schedule $p_i \in \mathcal{P}_i$, independently of and simultaneously to the other firms, in the second stage the agent observes the price vector $\mathbf{p} = (p_1, ..., p_n) \in \mathcal{P}_1 \times \cdots \times \mathcal{P}_n$, and selects a consumption set $\mathbf{S}(\mathbf{p}) \subseteq \Omega$ as a function of $\mathbf{p}$. Thus, the set of strategies of the agent is the set of functions $\mathbf{S}(\mathbf{p})$ from $\mathcal{P}_1 \times \cdots \times \mathcal{P}_n$ to $2^{\Omega_1} \times \ldots \times 2^{\Omega_n}$. Hence, formally, we have a strategic game with $n+1$ players, n firms and an agent. Let $G^{MB}(n+1, v, c)$ (where MB stands for mixed bundling pricing) denote such a game.

Finally, the payoff of each firm $i \in N$ is given by its profit function

$$\pi_i(\mathbf{S}(\mathbf{p})) = (p_i - c_i)(S_i(\mathbf{p})) \tag{1}$$

where $(p_i - c_i)(S_i(\mathbf{p}))$ means $p_i(S_i(\mathbf{p})) - c_i(S_i(\mathbf{p}))$. When there is not ambiguity we will write $\mathbf{S}$ instead of $\mathbf{S}(\mathbf{p})$, so that the firm i's profit is $\pi_i(\mathbf{S}) = (p_i - c_i)(S_i)$. The payoff for the agent when purchasing $\mathbf{S}$ at prices $\mathbf{p}$ is her consumer surplus,

$$cs[\mathbf{S}, \mathbf{p}] = v(\mathbf{S}(\mathbf{p})) - \sum_{i \in N} p_i(S_i(\mathbf{p})) = v(\mathbf{S}) - \sum_{i \in F(\mathbf{S})} p_i(S_i). \tag{2}$$

Since the prices set by each firm affect the profits of the other firms and the agent chooses whether to purchase from either all the firms, a subset of them or not purchasing at all, our model can be understood as a delegated agency game with contractual externalities, where n (multiproduct) principals offer contracts (price schedules) for any bundle of her own products and the agent (the consumer) chooses whether to accept all the contracts, a subset of them or none at all. Given the set of the agent's strategies, she is considered as another player and therefore game $G^{MB}(n+1, v, c)$ denotes such a delegated agency game, where the principals offer mixed-bundling contracts. Therefore, the strategy space of principals are the set of firms' strategies in the complete information game of firms and the consumer. Thus, we consider the extended space of contract schedules instead of only the equilibrium offers. Since we are dealing with a two-stage game of complete information, it seems appropriate to employ the solution concept of sub-game perfect Nash equilibrium.

A *subgame perfect Nash equilibrium* is a list of strategies, $(\widetilde{\mathbf{S}}, \widetilde{p}_1, \ldots, \widetilde{p}_n)$ one for each player, such that:

$$\widetilde{\mathbf{S}} \in \arg\max_{\mathbf{S}\subseteq\Omega} cs[\mathbf{S}, \widetilde{\mathbf{p}}] = \arg\max_{\mathbf{S}\subseteq\Omega} v(\mathbf{S}) - \sum_{i\in F(\mathbf{S})} \widetilde{p}_i(S_i) \tag{3}$$

and

$$\widetilde{p}_i \in \arg\max_{p_i\in\mathcal{P}_i} \pi_i(\widetilde{\mathbf{S}}(\widetilde{\mathbf{p}}_{-i}, p_i)) \text{ for all } i \in N \tag{4}$$

where $(\widetilde{\mathbf{p}}_{-i}, p_i) = (\widetilde{p}_1, \ldots, \widetilde{p}_{i-1}, p_i, \widetilde{p}_{i+1}, \ldots, \widetilde{p}_n)$.

Let *SPE* be the set of pure strategy subgame perfect equilibria of $G^{MB}(n+1, v, c)$. If $(\mathbf{S}, \mathbf{p})$ is an element in *SPE*, $\mathbf{p}$ is called an *SPE*-price vector, $\mathbf{S}$ is an *SPE*-consumption set and $(\mathbf{S}, \mathbf{p})$ is denoted as an *SPE*-outcome. If bundle $\mathbf{S}$ has goods of only a subset of firms, then the *SPE*-consumption set is a *partial common agency* allocation and the *SPE*-outcome is a *partial common agency equilibrium*, being a *common agency equilibrium* if bundle $\mathbf{S}$ has goods of all the firms. Alternatively, when $\mathbf{S}$ has goods of only one firm, then the *SPE*-consumption set is an *exclusive dealing* allocation and the *SPE*-outcome is an *exclusive dealing equilibrium*.

Let function $(v-c)(\mathbf{S}) = v(\mathbf{S}) - \sum_{i\in F(\mathbf{S})} c_i(S_i)$ be the *social surplus function*. Bundle $\widetilde{\mathbf{S}}$ is *socially efficient* if $\widetilde{\mathbf{S}} \in \arg\max_{\mathbf{S}} (v-c)(\mathbf{S})$. It is assumed that the maximized social surplus function is always positive[5]. An *SPE*-outcome is *efficient* if its *SPE*-consumption set maximizes the social surplus.

Two questions deserve some clarification. The first one is concerned to the use of mixed bundling strategies as opposed to linear prices. Linear contracts do not guarantee in these settings, even under extended contract schedules, the existence of (subgame perfect) equilibrium outcomes, either in common agency or in exclusive dealing (see Liao and Urbano (2002), for technical details). The intuition is clear: suppose a market for systems, where two firms produce two goods each and the agent has preferences over systems (all the bundles of two goods) and that common agency is the efficient allocation. Starting from an efficient allocation where the agent buys from the two principals, any of them may find it profitable to deviate and exclusively deal with the agent. However, these deviations need not lead to an exclusive dealing equilibrium because both principals will compete fiercely and a deviation from a principal may be followed by other deviations from the rival and the sequence of deviations need not converge.[6] Example 1 below shows that linear prices do not guarantee the existence of *SPE*-outcomes. The second concerns refers to considering

[5] Otherwise, if for every consumption set S, $v(S) < \sum_{i\in F(S)} c_i(S_i)$, then the model is degenerated. Hence, at every equilibrium point (S, p), $S = (\emptyset, \ldots, \emptyset)$ must hold and therefore no production will take place.

[6] Alternatively, consider that an exclusive dealing bundle is the efficient allocation and that the common agency bundles are quite attractive to the agent as compared with the exclusive dealing ones. In this case, the other exclusive dealing allocation is not very attractive to the agent and then, because of competition, the agent may get most of the surplus. Then, the principal selling the efficient bundle may find it profitable to raise the price of one of the individual components of its own bundle and set the other one in such a way that the agent chooses now the common agency bundle thus giving more profits to this principal. Again,

the extended space of contract schedules instead of only the equilibrium offers. Namely, as in Chiesa and Denicolò (2009) the subgame perfect equilibrium entails the inclusion of out-of-equilibrium offers. Example 1 (continuation) shows that equilibrium supply schedules must contain some contracts that will never be accepted. In Section 4 we will discuss how many of such contracts are needed to support an equilibrium.

Example 1 *(The market for systems)*: Linear Prices. Let the set of firms (principals) be $N = \{1, 2\}$, producing $\Omega_1 = \{a, b\}$ and $\Omega_2 = \{c, d\}$, respectively. Assume for simplicity that $c_i(w) = 0$ for all $i \in N$ and $w \in \Omega_i$ and that firms set linear prices, i.e. the exclusive dealing bundles $\{a, b\}$ and $\{c, d\}$ are sold at prices $p_a + p_b$, and $p_c + p_d$, respectively. The agent's value function is,

$$v(S) = \begin{cases} 4 & S = \{a, b\} \\ 9 & S = \{a, d\} \\ 5 & S = \{b, c\} \\ \delta & S = \{c, d\} \\ 0 & \text{otherwise} \end{cases}$$

with $0 < \delta < 9$.[7]

The efficient consumption set is the common agency bundle $\mathbf{S} = \{a, d\}$. Suppose that $\delta = 8$. In the Appendix it is shown that no linear prices support $\mathbf{S} = \{a, d\}$ as the equilibrium consumption set. The principals' incentives to deviate and exclusively deal with the agent cannot be overcame by linear pricing. The intuition is that every time the price of a good in a bundle is changed, the prices of the other bundles which contain that good are also changed. Firms can use this property to design profitable deviations by either increasing or decreasing subsets of prices to discourage the consumption of the (efficient) common agency consumption set while encouraging that of exclusive dealing. The above reasoning is applied to any other bundle to conclude that *under linear pricing the equilibrium may fail to exist.*

Given the above result, one question is whether another kind of extended contracts - mixed bundling prices- ensures the existence of equilibrium. The answer is affirmative. Under mixed bundling contracts the agent has the option of buying bundles of goods from a firm at a discount over the single good prices. Hence, in the above market for systems example, mixed bundling contracts are conditional on exclusive dealing. In a more general model, with multi-product firms and an agent with preferences over any bundle of goods, mixed bundling contracts can be conditional on exclusive dealing for each bundle of two or more goods. Therefore, mixed-bundling contracts can be seen as either an aggressive pricing policy for exclusive dealing outcomes or as out-of-equilibrium offers sustaining the

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this deviation need not lead to a delegated common agency equilibrium since the other principal may find another profitable deviation and so on.

[7] Notice that a more rigorous notation would be to denote by $\{\{a, b\}, \varnothing\}$ and $\{\varnothing, \{c, d\}\}$ the one-firm bundles and by $\{\{a\}, \{d\}\}$ and $\{\{b\}, \{c\}\}$ the two-firm bundles. To ease the notation and since no confusion will arise we follow the more simple notation of $\{a, b\}$, $\{c, d\}$, $\{a, d\}$ and $\{b, c\}$, to respectively denote such sets.

equilibrium consumption sets of individual components in delegated common agency allocations (involving either several principals or all of them). The discrimination on exclusivity helps principals set incentive-compatible contracts by both facilitating collusion on common agency outcomes and by representing a credible threat that avoids deviations by the principals. Thus, mixed bundling contracts makes it easier to sustain equilibrium outcomes and are sufficient to guarantee the existence of equilibrium

Example 1 (continuation): Mixed bundling prices and the role of out of equilibrium offers. Consider example 1 where $\delta = 8$. Notice that in this case the common agency efficient bundle $\mathbf{S} = \{a, d\}$ is not highly valued by the agent, in the sense that $v(a, d) < v(a, b) + v(c, d)$. This will imply that the agent will get some surplus. Suppose that firms do not precommit to linear pricing and let p_{ab} and p_{cd} be the prices of the exclusive dealing bundles $\{a, b\}$ and $\{c, d\}$, respectively, then $p_{ab} < p_a + p_b$, and $p_{cd} < p_c + p_d$.

Since the efficient bundle $\mathbf{S}$ is one of common agency some particularly important deviations by principals are those to their exclusive dealing bundles. Although mixed-bundling contracts discriminate on exclusivity, if the out-of equilibrium prices p_{ab} and p_{cd} are priced at unit costs, they become quite attractive to the agent and then the prices of the goods of the efficient common agency bundle $\mathbf{S}$ have to be quite low to avoid the agent's choice of an exclusive dealing bundle. Therefore, there is an incentive for each firm to increase the prices of its own bundles above unit costs, making them less attractive. However, equilibrium subgame perfection imposes some restrictions on firms' exclusive dealing prices. In particular, each firm will set the price of its own bundle as to guarantee itself the same profit margin as under $\mathbf{S}$ and then the equilibrium price vector will have to be immune to such an action.

In the Appendix we show that the efficient bundle $\mathbf{S} = \{a, d\}$ is a *SPE*-consumption set, supported by equilibrium prices verifying:

$$\begin{aligned}
&0 \le p_a \le 1,\ 4 \le p_d \le 5,\ 5 \le p_a + p_d,\\
&p_{ab} = p_a + p_d - 5,\ p_{cd} = p_a + p_d - 1,\\
&p_b \text{ and } p_c \text{ bigger enough.}
\end{aligned}$$

Thus, the out-of-equilibrium exclusive dealing prices p_{ab} and p_{cd} are set strategically by firms to make the exclusive dealing bundles as profitable for the consumer as the efficient bundle, and the prices for products b and c are set high enough to make the other common agency bundle $\{b, c\}$ unprofitable for the agent. Notice also that $p_{ab} \le p_a$ and $p_{cd} \le p_d$ and that the consumer surplus is positive and upper bounded, $3 \le cs[\mathbf{S}, \mathbf{p}] = 9 - p_a - p_d \le 4$.

This example also illustrates that *principals submitting only their efficient contract and the null contract, i.e., making a take-it-or leave-it (TIOLI) offer, need not be a subgame perfect equilibrium.*

3. Subgame perfect equilibrium and efficiency: Strong Equilibria

In what follows, we characterize both the set of subgame perfect equilibria and the set of efficient subgame perfect equilibria of the delegated agency game $G^{MB}(n + 1, v, c)$ where the principals (firms) might use mixed bundling contracts.

It is standard in agency games[8] that every principal i plays a bargaining game with the agent and hence leaves her with a payoff that does not exceed her disagreement payoff, which is the maximum payoff she can obtain by trading optimality with the remaining $N \setminus \{i\}$ principals (*individual excludability*). This means that given the $N \setminus \{i\}$ principals' equilibrium strategies, the agent equilibrium payoff does nor depend of firm i's strategy. As a consequence, the strategy of firm i must maximize his joint payoff with the agent, given the other principals' equilibrium strategies (*bilateral efficiency*). Individual excludability and bilateral efficiency characterize the equilibrium of the agency game in Chiesa and Denicolò (2009, see Lemma 1, pag. 301). This characterization result recurs in the common agency literature (see Bernheim and Whinston (1986a, Lemma 2) and the fundamental equations of Laussel and Le Breton (2001)).

The next Proposition characterizes the set of *SPE*-outcomes[9] of our agency game and parallels the aforementioned Lemma 1 in Chiesa and Denicolò (2009). This characterization is equivalent to the translation of individual excludability and bilateral efficiency properties to our setting and in terms of principals' deviations. Namely, condition BC below precludes unilateral deviations by the agent. Condition FC1 refers to individual excludability and guarantees that each active principal j does not have an incentive to increase the equilibrium prices of its sold bundles (to increase his payoff), since there is at least a bundle of the other principals that leave the agent with the same equilibrium payoff. Bilateral efficiency or pairwise stability is given by conditions FC2 and FC3. Since principals are multiproduct firms selling bundles of products and at equilibrium they sell only one, condition FC2 says that each active principal j does not have an incentive to reduce the prices of her unsold bundles to those of the sold ones in order to sell any of them profitably, given the other principals' equilibrium strategies. On the other hand, condition FC3 additionally guarantees that each non-active principal cannot benefit from price reductions to unit costs. These two conditions are equivalent to saying that the *equilibrium joint payoff of the agent and a principal is the maximum joint payoff given the other principals' equilibrium strategies.*

Proposition 1. $(\widetilde{\mathbf{S}}, \widetilde{\mathbf{p}})$ *is an* SPE*-outcome of* G^{MB}*, where* $\widetilde{\mathbf{S}} = (\widetilde{S}_1, \ldots, \widetilde{S}_n) \subseteq \Omega$ *and* $\widetilde{\mathbf{p}} = (\widetilde{p}_1, ..., \widetilde{p}_n)$*,* $\widetilde{p}_i \in \mathcal{P}_i$ *with* $\widetilde{p} \geq c$*, if and only if*

For all $\mathbf{S} \subseteq \Omega$, $cs[\widetilde{\mathbf{S}}, \widetilde{\mathbf{p}}] \geq cs[\mathbf{S}, \widetilde{\mathbf{p}}]$*, i.e.,*

$$v(\widetilde{\mathbf{S}}) - \sum_{i \in F(\widetilde{\mathbf{S}})} \widetilde{p}_i(\widetilde{S}_i) \geq v(\mathbf{S}) - \sum_{i \in F(\mathbf{S})} \widetilde{p}_i(S_i). \tag{BC}$$

For every $j \in F(\widetilde{\mathbf{S}})$ *there is* $\mathbf{S}^j \subseteq \Omega$ *with* $S^j_j = \emptyset$ *such that,* $cs[\widetilde{\mathbf{S}}, \widetilde{\mathbf{p}}] = cs[\mathbf{S}^j, \widetilde{\mathbf{p}}]$*, i.e.,*

[8] Where principals sell a homogeneous, perfectly divisible product and the agent's utility function is strictly concave.

[9] These conditions extend those of Arribas and Urbano (2005) and Tauman *et al.* (1997) to multiproduct principals.

$$v(\widetilde{\mathbf{S}}) - \sum_{i \in F(\widetilde{\mathbf{S}})} \widetilde{p}_i(\widetilde{S}_i) = v(\mathbf{S}^j) - \sum_{i \in F(\mathbf{S}^j)} \widetilde{p}_i(S_i^j). \tag{FC1}$$

For each $j \in F(\widetilde{\mathbf{S}})$ *and all* $\mathbf{S} \subseteq \Omega$ *such that* $j \in F(\mathbf{S})$, $cs[\widetilde{\mathbf{S}}, \widetilde{\mathbf{p}}] \geq cs[\mathbf{S}, (\widetilde{\mathbf{p}}_{-j}, \widetilde{p}_j(\widetilde{S}_j) - c_j(\widetilde{S}_j) + c_j(S_j))]$, *i.e.,*

$$v(\widetilde{\mathbf{S}}) - \sum_{i \in F(\widetilde{\mathbf{S}})} \widetilde{p}_i(\widetilde{S}_i) \geq v(\mathbf{S}) - [\widetilde{p}_j(\widetilde{S}_j) - c_j(\widetilde{S}_j) + c_j(S_j)] - \sum_{i \in F(\mathbf{S}) \setminus j} \widetilde{p}_i(S_i). \tag{FC2}$$

For each $j \notin F(\widetilde{\mathbf{S}})$ *and for all* $\mathbf{S} \subseteq \Omega$ *such that* $j \in F(\mathbf{S})$, $cs[\widetilde{\mathbf{S}}, \widetilde{\mathbf{p}}] \geq cs[\mathbf{S}, (\widetilde{\mathbf{p}}_{-j}, c_j(S_j))]$, *i.e.,*

$$v(\widetilde{\mathbf{S}}) - \sum_{i \in F(\widetilde{\mathbf{S}})} \widetilde{p}_i(\widetilde{S}_i) \geq v(\mathbf{S}) - c_j(S_j) - \sum_{i \in F(\mathbf{S}) \setminus j} \widetilde{p}_i(S_i). \tag{FC3}$$

Notice that BC-FC3 are implied by subgame perfection requirements: BC by the agent and (FC1-FC3 by the firms' incentives. To see this, suppose that FC1 does not hold, then by BC there is $j \in N$, such that for all $\mathbf{S}^j \subseteq \Omega$ with $S_j^j = \emptyset$,

$$v(\widetilde{\mathbf{S}}) - \sum_{i \in F(\widetilde{\mathbf{S}})} \widetilde{p}_i(\widetilde{S}_i) > v(\mathbf{S}^j) - \sum_{i \in F(\mathbf{S}^j)} \widetilde{p}_i(S_i^j).$$

The above inequality implies that firm j is better off charging a price $\widetilde{p}_j(\widetilde{S}_j) + \varepsilon$, for a sufficiently small $\varepsilon > 0$, such that BC is still satisfied. Now, the agent observing the new price vector will again choose the consumption set $\widetilde{\mathbf{S}}$, but firm j will obtain an extra profit of ε. Hence FC1 must be verified if $(\widetilde{\mathbf{S}}, \widetilde{\mathbf{p}})$ is an *SPE*-outcome.

If FC2 does not hold, then for some firm $j \in N$ there is a consumption set $\mathbf{S} \subseteq \Omega$ such that.

$$v(\widetilde{\mathbf{S}}) - \sum_{i \in F(\widetilde{\mathbf{S}})} \widetilde{p}_i(\widetilde{S}_i) < v(\mathbf{S}) - [\widetilde{p}_j(\widetilde{S}_j) - c_j(\widetilde{S}_j) + c_j(S_j)] - \sum_{i \in F(\mathbf{S}) \setminus j} \widetilde{p}_i(S_i).$$

Hence firm j can set a price $p_j(S_j) = \widetilde{p}_j(\widetilde{S}_j) - c_j(\widetilde{S}_j) + c_j(S_j) + \varepsilon$, for a sufficiently small $\varepsilon > 0$, such that still

$$v(\widetilde{\mathbf{S}}) - \sum_{i \in F(\widetilde{\mathbf{S}})} \widetilde{p}_i(\widetilde{S}_i) < v(\mathbf{S}) - p_j(S_j) - \sum_{i \in F(\mathbf{S}) \setminus j} \widetilde{p}_i(S_i),$$

which implies that the agent will now select the consumption set $\mathbf{S}$ and firm j will increase its profits.

Finally, if FC3 is not verified, then for some firm $j \notin F(\widetilde{\mathbf{S}})$ there is a consumption set $\mathbf{S} \subseteq \Omega$ with $j \in F(S)$ such that,

$$v(\widetilde{\mathbf{S}}) - \sum_{i \in F(\widetilde{\mathbf{S}})} \widetilde{p}_i(\widetilde{S}_i) < v(\mathbf{S}) - c_j(S_j) - \sum_{i \in F(\mathbf{S}) \setminus j} \widetilde{p}_i(S_i).$$

Thus, similarly to the above, if firm j sets price $p_j(S_j) = c_j(S_j) + \varepsilon$, for a sufficiently small $\varepsilon > 0$, then the agent will select set $\mathbf{S}$ and firm j will increase its profits.

Conversely, if BC, FC1, FC2 and FC3 are satisfied, then $(\widetilde{\mathbf{S}}, \widetilde{\mathbf{p}})$ is an *SPE*-outcome since $\widetilde{\mathbf{S}}$ is a best choice for the agent and no firm has an incentive to either reduce or increase its prices. Notice that the set $\mathbf{S}^j$ in FC1 may be empty and in this case $cs[\widetilde{\mathbf{S}}, \widetilde{\mathbf{p}}] = 0$, and firms extract the entire consumer surplus.

In agency model where the agent's utility function is strictly concave and principals sell a homogeneous product, then the social surplus function is globally concave and the subgame perfect Nash-equilibriun outcome is efficient, i.e., it maximizes the social surplus function. Thus, sub-game perfection implies efficiency. In contrast, in our model with multiproduct firms and where the agent's utility function over bundles need not be strictly concave, the Nash concept of stability (or bilateral efficiency) may not be enough to guarantee efficiency. This is particularly true when the equilibrium bundle pertains to several firms so that additional coordination between principals may be required. Here, sub-game perfection does not rule out inefficient equilibria and therefore the conditions of Proposition 1 characterize both efficient and inefficient subgame perfect Nash-equilibrium as the following example illustrates.

Example 2. Efficient and inefficient equilibrium outcomes: Consider again example 1, now with $0 < \delta < 1$. In this case both common agency bundles $\{a, d\}$ and $\{b, c\}$ are highly valuable to the agent in the sense that $v(a, d) > v(b, c) > v(a, b) + v(c, d)$. We prove that now both efficient and inefficient equilibrium outcomes exist, where the consumer gets zero surplus and all the rent goes to principals.

By the same reasoning than in example 1 (continuation) above, p_{ab} and p_{cd} cannot be equal to unit costs. However, since the two common agency bundles are highly valued with respect to the exclusive dealing ones, subgame perfection translates to setting $p_{ab} = M > v(a, b)$, $p_{cd} = M' > v(c, d)$ and then the binding constraint is $p_a + p_d = v(a, d) = 9 > v(a, b) + v(c, d)$, with $p_a \geq v(a, b)$ and $p_d \geq v(c, d)$. This additionally avoids principals' price deviation in order to sell its exclusive dealing bundle.

The high prices of products b and c and those of the exclusive dealing bundles $\{a, b\}$ and $\{c, d\}$ make it only attractive for the agent the efficient common agency bundle $\mathbf{S} = \{a, d\}$. The agent's surplus is zero and firm 1's profit is $9 - p_d$. But principal 1 could sell consumption set $\{a, b\}$ instead by setting $p_a = p_b = M$ (M big enough) and $p_{ab} = 4$. Hence, at the equilibrium it must be that $9 - p_d \geq 4$ or $p_d \leq 5$; similarly, in order firm 2 does not deviate,

$p_a \leq 9 - \delta$. Hence, at the equilibrium $p_a + p_d = 9$ with $4 \leq p_a \leq 9 - \delta$ and $\delta \leq p_d \leq 5$.

Thus, a sub-game perfect equilibrium is the *efficient common agency outcome* with consumption set $\mathbf{S} = \{a, d\}$ and equilibrium prices satisfying:

$$\begin{aligned} p_a + p_d &= 9,\ 4 \leq p_a \leq 9 - \delta,\ \delta \leq p_d \leq 5, \text{ and} \\ p_k &= M, k \in \{b, c, \{a, b\}, \{c, d\}\}. \end{aligned}$$

The efficient common agency bundle is very valued by the agent compared with any of the exclusive dealing bundles. Hence, principals can obtain all the surplus. Notice that there are multiple sub-game perfect equilibrium prices sustaining the efficient common agency consumption set but in all of them the agent's surplus is zero and the sum of the firms' profits is a constant. Among them, let us highlight the following ones:

$$\begin{aligned} &p_a = 9 - \delta, p_d = \delta, p_b = p_c = p_{ab} = p_{cd} = M, \\ &p_a = 4, \qquad p_d = 5, p_b = p_c = p_{ab} = p_{cd} = M, \end{aligned}$$

where the first one gives the maximum possible profits to principal 1 and the second one does the same for principal 2. Any convex combination of these price vectors is also an equilibrium price vector.

Nevertheless, it is also easy to show that the *inefficient common agency outcome* $\{b, c\}$, supported by the price vector: $p_b + p_c = v(b, c) = 5$; $p_b \geq v(a, b) = 4$; $p_c \geq v(c, d) = \delta$; $p_k = M, \quad k \in \{a, d, \{a, b\}, \{c, d\}\}$, is also a sub-game perfect equilibrium. The reason parallels the one above. There are also multiple sub-game perfect equilibrium prices sustaining the inefficient consumption set. For instance,

$$\begin{aligned} &p_b = 5 - \delta, p_c = \delta, p_a = p_d = p_{ab} = p_{cd} = M, \text{ or} \\ &p_b = 4, \qquad p_c = 1, p_a = p_d = p_{ab} = p_{cd} = M. \end{aligned}$$

Notice that the inefficient outcome is not perturbed even if either principal 1 reduces the price of its remaining unsold goods to the price of the sold one, i.e. $p_{ab} = p_a = p_b$ or principal 2 sets $p_{cd} = p_d = p_c$. It can be easily checked that all the *SPE*-outcomes satisfy conditions BC to FC3.

In conclusion, Proposition 1 characterizes the set of all *SPE*-outcomes, but both efficient and inefficient outcomes belong to the Nash equilibrium correspondence.

However, if in the example 2 both firms coordinate and make a price reduction simultaneously then the agent will have an incentive to buy a different bundle. The problem with these inefficient equilibria is that the two principals charge high prices for the goods of the efficient common agency bundle so that no individual firm can benefit from a price reduction of its part of the efficient bundle. Therefore a higher degree of coordination among principals' is needed to achieve efficiency. To isolate efficient *SPE*-outcomes from inefficient ones, a new condition has to be imposed as illustrated below.

Example 2. (continuation) Suppose that in example 2 above *both* active principals *simultaneously* reduce the prices of their unsold goods to those of the sold ones at a given inefficient *SPE*. For instance, consider the inefficient *SPE*-outcome, $\{b, c\}$, and prices: $p_b = 5 - \delta, p_c = \delta, p_a = p_d = p_{ab} = p_{cd} = M$. Now, let be $p'_a = p'_b = p'_{ab} = 5 - \delta$, the price of the sold product b; and let be $p'_c = p'_d = p'_{cd} = \delta$, the price of the sold product c. At these new prices:

$$cs[\{a, d\}, \mathbf{p}'] = v(a, d) - (p'_a + p'_d) = 9 - (5 - \delta + \delta) = 4 > 0 = v(b, c) - (p'_b + p'_c) = cs[\{b, c\}, \mathbf{p}']$$

and the inefficient *SPE* is ruled out. The same reasoning rules out all the remaining inefficient equilibria. Furthermore, it is also easily checked that all the efficient *SPE*'s are immune to these simultaneous price reductions.

With this idea in mind, we would like to consider only the subset of subgame perfect equilibrium outcomes which remains as equilibrium outcomes even if *all* non-active principals set unit cost prices and *all* active principals set prices for their unsold bundles equal to those of their sold ones adjusted by the cost-differential.[10] In other words, we want FC3 to be satisfied for all subsets of non-active firms: for all $A \subseteq N \backslash F(\widetilde{\mathbf{S}})$ and FC2 to be satisfied for all subsets of active firms: for all $B \subseteq F(\widetilde{\mathbf{S}})$. Thus, the condition of bilateral efficiency or pairwise stability has to be extended to a notion of stability against joint deviations that are mutually profitable to any subset of principals and the agent. We denote these conditions as *strong efficiency* or *strong stability*, meaning that the equilibrium joint payoff of the agent and any subset of principals selling at equilibrium is the maximum joint payoff given all possible strategies of all other subsets of principals, even of those subsets not selling at equilibrium.[11] These conditions remove the set of *SPE*-outcomes in which some firms charge unreasonably high prices so that no individual firm can benefit from a price reduction of its products only.

To define these restrictions on set *SPE*, consider the price vector $\mathbf{p} = (p_1, ..., p_n) \in \mathcal{P}_1 \times \cdots \times \mathcal{P}_n$, and let $\mathbf{S} \subseteq \Omega$. Define vector $\mathbf{p}^{\mathbf{S}}$ for all $i \in N$, $T_i \subseteq \Omega_i$, as

$$p_i^{\mathbf{S}}(T_i) = \begin{cases} p_i(S_i) & \text{if } i \in F(\mathbf{S}),\ T_i = S_i \\ p_i(S_i) - c_i(S_i) + c_i(T_i) & \text{if } i \in F(\mathbf{S}),\ T_i \neq S_i \\ c_i(T_i) & \text{if } i \notin F(\mathbf{S}), \end{cases} \tag{5}$$

i.e. all the non-active firms set prices equal to the marginal cost, and all active firms set prices for unsold bundles equal to those of their sold bundles adjusted by the cost-differentials.

Definition 1. *For every triple (N, v, c), the subset* SPE* *of* SPE-*outcomes of* G^{MB} *is defined as:*

$$SPE^* = \{(\mathbf{S}, \mathbf{p}) \in SPE | (\mathbf{S}, \mathbf{p}^{\mathbf{S}}) \in SPE\}.$$

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[10]Notice that if unit costs were assumed to be zero, then these prices would amount to being equal to those of the sold bundles.

[11]An equivalent definiton could be that of coalition-proof efficiency or coalition-proff stability. We do not use it, because it could be related to the concept of coalition-proof equilibrium, where only self-enforcing deviations are considered to be credible threats. A deviation by a coalition is sel-enforcing if no sub-coalition has an incentive to iniciate a new deviation.

Equivalently, *SPE** is the set of equilibrium outcomes satisfying BC, FC1, and FC4 (instead of FC2 and FC3), where FC4 is stated as:

For all $A \subseteq N \backslash F(\widetilde{\mathbf{S}})$, $B \subseteq F(\widetilde{\mathbf{S}})$ and for all $\mathbf{S} \subseteq \Omega$ such that $(A \cup B) \subseteq F(\mathbf{S})$,

$$cs[\widetilde{\mathbf{S}}, \widetilde{\mathbf{p}}] \geq cs[\mathbf{S}, ((\widetilde{p}_i)_{i \in F(\mathbf{S}) \backslash (A \cup B)}, (c_i(S_i))_{i \in A}, (\widetilde{p}_i(\widetilde{S}_i) - c_i(\widetilde{S}_i) + c_i(S_i))_{i \in B})],$$

i.e.,

$$v(\widetilde{\mathbf{S}}) - \sum_{i \in F(\widetilde{\mathbf{S}})} \widetilde{p}_i(\widetilde{S}_i) \geq v(\mathbf{S}) - \sum_{i \in F(\mathbf{S}) \backslash (A \cup B)} \widetilde{p}_i(S_i) - \sum_{i \in A} c_i(S_i) - \sum_{i \in B} [\widetilde{p}_i(\widetilde{S}_i) - c_i(\widetilde{S}_i) + c_i(S_i)]. \quad \text{(FC4)}$$

Thus, we restrict the analysis to a certain subset *SPE** of *SPE*-outcomes. The next Proposition shows that FC4 selects the set of efficient consumption bundles of G^{MB}.

Proposition 2. *For every value function v and unit cost vector c, if $(\widetilde{\mathbf{S}}, \widetilde{\mathbf{p}})$ is an* SPE*-*outcome of G^{MB}, then $\widetilde{\mathbf{S}}$ is socially efficient.*

Proof: If $(\widetilde{\mathbf{S}}, \widetilde{\mathbf{p}})$ is an *SPE**-outcome, then $(\widetilde{\mathbf{S}}, \widetilde{\mathbf{p}}^{\widetilde{\mathbf{S}}}) \in SPE^*$. By BC in Proposition 1, $cs[\widetilde{\mathbf{S}}, \widetilde{\mathbf{p}}^{\widetilde{\mathbf{S}}}] \geq cs[\mathbf{S}, \widetilde{\mathbf{p}}^{\widetilde{\mathbf{S}}}]$ for all $\mathbf{S} \subseteq \Omega$. Therefore,

$$\begin{aligned} (v - c)(\widetilde{\mathbf{S}}) - (v - c)(\mathbf{S}) &\geq \sum_{i \in F(\widetilde{\mathbf{S}})} (\widetilde{p}_i^{\widetilde{\mathbf{S}}} - c_i)(\widetilde{S}_i) - \sum_{i \in F(\mathbf{S})} (\widetilde{p}_i^{\widetilde{\mathbf{S}}} - c_i)(S_i) \\ &= \sum_{i \in F(\widetilde{\mathbf{S}})} (\widetilde{p}_i^{\widetilde{\mathbf{S}}} - c_i)(\widetilde{S}_i) - \sum_{i \in F(\widetilde{\mathbf{S}}) \cap F(\mathbf{S})} (\widetilde{p}_i^{\widetilde{\mathbf{S}}} - c_i)(S_i) \\ &= \sum_{i \in F(\widetilde{\mathbf{S}}) \backslash F(\mathbf{S})} (\widetilde{p}_i - c_i)(\widetilde{S}_i) \geq 0, \end{aligned}$$

given that $\widetilde{p}_i^{\widetilde{\mathbf{S}}}(S_i) = c_i(S_i)$ for all $i \notin F(\widetilde{\mathbf{S}})$ and $p_i^{\widetilde{\mathbf{S}}}(S_i) - c_i(S_i) = p_i^{\widetilde{\mathbf{S}}}(\widetilde{S}_i) - c_i(\widetilde{S}_i)$ for all $i \in F(\widetilde{\mathbf{S}}) \cap F(\mathbf{S})$. Thus, $(v - c)(\widetilde{\mathbf{S}}) \geq (v - c)(\mathbf{S})$ for every $\mathbf{S} \subseteq \Omega$. ■

The definition of *SPE** captures the idea of equilibrium stability against joint deviations that are mutually profitable to any subset of firms and the agent. We look for an equilibrium concept behind conditions BC, FC1, and FC4, or set *SPE**, refining the set of *SPE*-outcomes. Aumann (1959) proposed the concept of *Strong Equilibrium* as an equilibrium such that no subset of players has a *joint* deviation that strictly benefits *all* of them. While the Nash concept of stability defines equilibrium only in terms of unilateral deviations, strong Nash equilibrium allows for deviations by every conceivable coalition. The extension of that concept to a *Subgame Perfect Strong Equilibrium –SPSE–* of our game G^{MB} is as follows,

Definition 2. $(\widehat{\mathbf{S}}, \widehat{\mathbf{p}})$ *is an* SPSE-*outcome of G^{MB}, where $\widehat{\mathbf{S}} = (\widehat{S}_1, \ldots, \widehat{S}_n) \subseteq \Omega$ and $\widehat{\mathbf{p}} = (\widehat{p}_1, ..., \widehat{p}_n)$, $\widehat{p}_i \in \mathcal{P}_i$ with $\widehat{\mathbf{p}} \geq c$ if*

i) $cs[\widehat{\mathbf{S}}, \widehat{\mathbf{p}}] \geq cs[\mathbf{S}, \widehat{\mathbf{p}}]$ for all $S \subseteq \Omega$,

ii) For all $M \subseteq N$ there is no $(p'_j)_{j \in M}$ and $\mathbf{S} \subseteq \Omega$ such that, *a)* $(p' - c)_j(S_j) > (\widehat{p} - c)_j(\widehat{S}_j)$*, for each* $j \in M$*, and*

b) $cs[\widehat{\mathbf{S}}, \widehat{\mathbf{p}}] < cs[\mathbf{S}, ((\widehat{p}_i)_{i\in N\setminus M}, (p'_i)_{i\in M})]$

Since the deviating coalition can be either an individual principal or the agent, this implies that an *SPSE*-outcome is therefore an *SPE*. Moreover, it is shown next that conditions BC, FC1, and FC4 are the only conditions characterizing *SPSE*-outcomes.

Proposition 3. *For every value function* v*, unit cost vector* c*, and game* G^{MB}*, the set of* SPSE*-outcomes coincides with the set of* SPE**-outcomes.*

Proof: Let us first prove that any *SPSE*-outcome is an *SPE**. Let $(\widehat{\mathbf{S}}, \widehat{\mathbf{p}})$ be an *SPSE* and suppose that it is not an *SPE**. By FC4 of the definition of *SPE**, there are sets $A \subseteq N\setminus F(\widehat{\mathbf{S}})$, $B \subseteq F(\widehat{\mathbf{S}})$ and $\mathbf{S} \subseteq \Omega$ such that,

$$v(\widehat{\mathbf{S}}) - \sum_{i\in F(\widehat{\mathbf{S}})} \widehat{p}_i(\widehat{S}_i) < v(\mathbf{S}) - \sum_{i\in F(\mathbf{S})\setminus(A\cup B)} \widehat{p}_i(S_i) - \sum_{i\in A} c_i(S_i) - \sum_{i\in B}[\widehat{p}_i(\widehat{S}_i) - c_i(\widehat{S}_i) + c_i(S_i)]$$

and the consumer switches to $\mathbf{S}$. Thus, the coalition of firms in $A \cup B$ has an incentive to jointly deviate, obtaining higher profits than those under $(\widehat{\mathbf{S}}, \widehat{\mathbf{p}})$, which contradicts $(\widehat{\mathbf{S}}, \widehat{\mathbf{p}})$ being an *SPSE*-outcome.

Now let us assume that $(\widetilde{\mathbf{S}}, \widetilde{\mathbf{p}})$ is an *SPE**-outcome, but not an *SPSE*-outcome. Since both equilibrium concepts are subgame perfect, then $(\widetilde{\mathbf{S}}, \widetilde{\mathbf{p}})$ would not verify condition *ii)* above: there exists $M \subseteq N$, $(p'_j)_{j\in M}$ and $\mathbf{S} \subseteq \Omega$ such that for all $j \in M$, $(p' - c)_j(S_j) > (\widetilde{p} - c)_j(\widetilde{S}_j)$, and $cs(\mathbf{S}, ((p_j)_{j\in N\setminus M}, (p'_j)_{j\in M})) > cs(\widetilde{\mathbf{S}}, \widetilde{\mathbf{p}})$. Define the sets $A = M \cap (N\setminus F(\mathbf{S}))$ and $B = M \cap F(\mathbf{S})$, then condition FC4 does not hold, which contradicts $(\widetilde{\mathbf{S}}, \widetilde{\mathbf{p}})$ being an *SPE**. ■

Therefore set *SPSE* coincides with set *SPE**, and hence any *SPSE*-consumption set is socially efficient.

4. Existence, efficiency and characterization of Subgame Perfect Strong Equilibria.

4.1. Existence and efficiency

Our main result establishes that given any agent's value function v, and unit cost vector c, there is always a subgame perfect equilibrium of the delegated agency game G^{MB} verifying conditions BC, FC1 and FC4, i.e. set *SPSE* is non-empty (Theorem 1). Moreover, not only any *SPSE*-consumption set is efficient (Proposition 2), but also any socially efficient consumption set belongs to an *SPSE*-outcome (Corollary 1).

Theorem 1. *For every value function,* v*, and unit cost vector,* c*, there is an equilibrium in* SPSE*.*

Proof: By Proposition 3, it suffices to prove that set SPE^* is non-empty. Let $\widetilde{\mathbf{S}} \in \arg\max_{\mathbf{S}} (v-c)(\mathbf{S})$. Define the set

$$\Pi^f = \{\boldsymbol{\pi} \in \mathbb{R}^n_+ | \pi_i = 0, \forall i \notin F(\widetilde{\mathbf{S}}) \text{ and } \sum_{i \in F(\widetilde{\mathbf{S}}) \setminus F(\mathbf{S})} \pi_i \leq (v-c)(\widetilde{\mathbf{S}}) - (v-c)(\mathbf{S}), \forall \mathbf{S} \subseteq \Omega\}.$$

Set Π^f is non-empty since $\mathbf{0} \in \Pi^f$ and it is bounded, since for every $\boldsymbol{\pi} \in \Pi^f$ we have $\pi_i \leq (v-c)(\widetilde{\mathbf{S}}) - (v-c)(\widetilde{\mathbf{S}} \setminus \widetilde{S}_i)$ for all $i \in F(\widetilde{\mathbf{S}})$, and $\pi_i = 0$ for all $i \in N \setminus F(\widetilde{\mathbf{S}})$.

Also observe that Π^f is closed and hence compact. Thus Π^f contains an element $\widetilde{\boldsymbol{\pi}}$ which is maximal with respect to the lexicographical order on Π^f. Let $\widetilde{\mathbf{p}} = (\widetilde{p}_1, ..., \widetilde{p}_n) \in \mathcal{P}_1 \times \cdots \times \mathcal{P}_n$ be defined as $\widetilde{p}_i(T_i) = \widetilde{\pi}_i + c_i(T_i)$ for all $i \in N, T_i \subseteq \Omega_i$. Notice that $\widetilde{p}_i(T_i) = c_i(T_i)$ for all $i \notin F(\widetilde{\mathbf{S}})$, and $\widetilde{p}_i(\widetilde{S}_i) - c_i(\widetilde{S}_i) = \widetilde{\pi}_i$ for all $i \in F(\widetilde{\mathbf{S}})$. This leads to,

$$\widetilde{p}_i(T_i) = \widetilde{\pi}_i + c_i(T_i) = \widetilde{p}_i(\widetilde{S}_i) - c_i(\widetilde{S}_i) + c_i(T_i),$$

for all $i \in F(\widetilde{\mathbf{S}})$ and $T_i \neq S_i$. Hence $\widetilde{p}_i(T_i) = \widetilde{p}_i^{\widetilde{\mathbf{S}}}(T_i)$ as defined in (5).

We claim that $(\widetilde{\mathbf{S}}, \widetilde{\mathbf{p}})$ is an SPE^*-outcome. Since $\widetilde{\mathbf{p}} = \widetilde{\mathbf{p}}^{\widetilde{\mathbf{S}}}$, it suffices to prove that $(\widetilde{\mathbf{S}}, \widetilde{\mathbf{p}})$ is an SPE-outcome. Let $\mathbf{S} \subseteq \Omega$, since $\widetilde{\boldsymbol{\pi}} \in \Pi^f$, then $\sum_{i \in F(\widetilde{\mathbf{S}}) \setminus F(\mathbf{S})} \widetilde{\pi}_i \leq (v-c)(\widetilde{\mathbf{S}}) - (v-c)(\mathbf{S})$ or equivalently

$$(v-c)(\widetilde{\mathbf{S}}) - \sum_{i \in F(\widetilde{\mathbf{S}})} \widetilde{\pi}_i \geq (v-c)(\mathbf{S}) - \sum_{i \in F(\mathbf{S})} \widetilde{\pi}_i.$$

hence,

$$v(\widetilde{\mathbf{S}}) - \sum_{i \in F(\widetilde{\mathbf{S}})} \left[\widetilde{\pi}_i + c(\widetilde{S}_i)\right] \geq v(\mathbf{S}) - \sum_{i \in F(\mathbf{S})} \left[\widetilde{\pi}_i + c(S_i)\right],$$

which by the definition of $\widetilde{p}_i$ is condition BC of Proposition 1.

Conditions FC2 and FC3 of Proposition 1 hold trivially, given that $\widetilde{p}_i(T_i) = \widetilde{p}_i^{\widetilde{\mathbf{S}}}(T_i)$.

To prove condition FC1 of Proposition 1 suppose, on the contrary, that exists $j \in F(\widetilde{\mathbf{S}})$ such that $v(\widetilde{\mathbf{S}}) - \sum_{i \in F(\widetilde{\mathbf{S}})} \widetilde{p}_i(\widetilde{S}_i) > v(\mathbf{S}) - \sum_{i \in F(\mathbf{S})} \widetilde{p}_i(S_i)$ for all $\mathbf{S} \subseteq \Omega$ with $S_j = \emptyset$. Let

$$\varepsilon = \min_{\mathbf{S} \subseteq \Omega, S_j = \emptyset} \left\{ v(\widetilde{\mathbf{S}}) - \sum_{i \in F(\widetilde{\mathbf{S}})} \widetilde{p}_i(\widetilde{S}_i) - \left(v(\mathbf{S}) - \sum_{i \in F(\mathbf{S})} \widetilde{p}_i(S_i) \right) \right\},$$

then $\varepsilon > 0$. Let $\widehat{\boldsymbol{\pi}} \in \mathbb{R}^n_+$ be defined as

$$\widehat{\pi}_i = \begin{cases} \widetilde{\pi}_i & i \neq j \\ \widetilde{\pi}_i + \varepsilon & i = j, \end{cases}$$

and let $\widehat{\mathbf{p}} = (\widehat{p}_1, ..., \widehat{p}_n)$ be defined as $\widehat{p}_i(T_i) = \widehat{\pi}_i + c_i(T_i)$ for all $i \in N, T_i \subseteq \Omega_i$. Notice that $\widehat{p}_i(T_i) = \widetilde{p}_i(T_i)$ for all $i \in N \backslash j$, $T_i \subseteq \Omega_i$ and $\widehat{p}_j(T_j) = \widetilde{p}_j(T_j) + \varepsilon$. Since $\widetilde{\boldsymbol{\pi}}$ is a maximal element of Π^f, then $\widehat{\boldsymbol{\pi}} \notin \Pi^f$. However, we will show that $\sum_{i \in F(\widetilde{\mathbf{S}}) \backslash F(\mathbf{S})} \widehat{\pi}_i \leq (v-c)(\widetilde{\mathbf{S}}) - (v-c)(\mathbf{S})$ for all $\mathbf{S} \subseteq \Omega$ and hence $\widehat{\boldsymbol{\pi}} \in \Pi^f$, which is a contradiction. Given $\mathbf{S} \subseteq \Omega$, if $j \in F(\mathbf{S})$, then $\sum_{i \in F(\widetilde{\mathbf{S}}) \backslash F(\mathbf{S})} \widehat{\pi}_i \leq (v-c)(\widetilde{\mathbf{S}}) - (v-c)(\mathbf{S})$ since $\widetilde{\pi}_i = \widehat{\pi}_i$ for $i \neq j$. Suppose next that $j \notin F(\mathbf{S})$, by the definition of ε

$$v(\widetilde{\mathbf{S}}) - \sum_{i \in F(\widetilde{\mathbf{S}})} \widetilde{p}_i(\widetilde{S}_i) - \varepsilon \geq v(\mathbf{S}) - \sum_{i \in F(\mathbf{S})} \widetilde{p}_i(S_i),$$

but by the definition of $\widehat{\mathbf{p}}$ and $\widetilde{\mathbf{p}}$ the left hand side of the above inequality can be written as,

$$v(\widetilde{\mathbf{S}}) - \sum_{i \in F(\widetilde{\mathbf{S}})} \widetilde{p}_i(\widetilde{S}_i) - \varepsilon = v(\widetilde{\mathbf{S}}) - \sum_{i \in F(\widetilde{\mathbf{S}})} \left[\widetilde{\pi}_i + c_i(\widetilde{S}_i)\right] - \varepsilon = (v-c)(\widetilde{\mathbf{S}}) - \sum_{i \in F(\widetilde{\mathbf{S}})} \widehat{\pi}_i,$$

and the right hand side as

$$v(\mathbf{S}) - \sum_{i \in F(\mathbf{S})} \widetilde{p}_i(S_i) = v(\mathbf{S}) - \sum_{i \in F(\mathbf{S})} [\widetilde{\pi}_i + c_i(S_i)] = (v-c)(\mathbf{S}) - \sum_{i \in F(\mathbf{S})} \widehat{\pi}_i.$$

Hence,

$$\sum_{i \in F(\widetilde{\mathbf{S}}) \backslash F(\mathbf{S})} \widehat{\pi}_i \leq (v-c)(\widetilde{\mathbf{S}}) - (v-c)(\mathbf{S}),$$

as claimed. Therefore $(\widetilde{\mathbf{S}}, \widetilde{\mathbf{p}})$ is an *SPE**-outcome. ■

The proof of Theorem 1 also shows that efficiency is a sufficient condition to belong to an *SPSE*-outcome, i.e. if $\widetilde{\mathbf{S}} \in \arg\max_{\mathbf{S}} (v-c)(\mathbf{S})$, then there is a price vector $\widetilde{\mathbf{p}}$ such that $(\widetilde{\mathbf{S}}, \widetilde{\mathbf{p}})$ is an *SPSE*-outcome. This, jointly with Proposition 2, allows us to assert that the *subgame perfect strong equilibrium* concept selects the set of efficient *SPE* of G^{MB} from the *SPE* correspondence.

Corollary 1. *For every value function v and unit cost vector c,* SPSE *is the set of efficient* SPE-*outcomes of G^{MB}.*

4.2. Characterization of SPSE-*profit vectors: the equilibrium principals' rents.*

SPSE-consumption sets have been characterized as the socially efficient ones. In this section, we characterize the set of firms' profits (the principals' rents) which comes from *SPSE*-outcomes in the delegated agency game G^{MB}. The characterization is made in terms of the core of the game and the marginal contribution of each principal. By Definition 1 and Proposition 3, notice that if $(\mathbf{S}, \mathbf{p})$ is an *SPSE*-outcome, then $(\mathbf{S}, \mathbf{p}^{\mathbf{S}})$ is also an *SPSE*-outcome and the principals and the agent obtain the same payoffs under such outcomes: the two equilibria are payoff-equivalent. Thus, any pair $(\mathbf{S}, \mathbf{p^S})$ makes it possible to identify its payoff-equivalence class, and for any *SPSE*-price vector we can only consider $(\mathbf{S}, \mathbf{p^S})$ as the representative element of its equivalence class. We will show that the vector of principals' profits from *SPSE*-outcomes are their most preferred points in the core of the agency game G^{MB}. First, let us define the *marginal contribution* of principal i as the difference between the maximum social surplus attainable, say $V^* = \max_{\mathbf{S}\subseteq\Omega} : (v-c)(\mathbf{S})$, and the maximum social surplus attainable when principal i is inactive, say V^*_{-i}, i.e.,

$$mc_i = V^* - V^*_{-i} = \max_{\mathbf{S}\subseteq\Omega} : (v-c)(\mathbf{S}) - \max_{\substack{\mathbf{S}\subseteq\Omega \\ S_i=\emptyset}} : (v-c)(\mathbf{S}). \tag{6}$$

Let us define the core as

$$\begin{aligned} &core(v-c) = \{(\pi^b, (\pi_i)_{i\in N}) \in \mathbb{R}^{n+1}_+ | \\ &\quad \pi^b + \sum_{i\in N} \pi_i = V(N) \text{ and } \pi^b + \sum_{i\in F(\mathbf{S})} \pi_i \geq (v-c)(\mathbf{S}),\ \forall S \subseteq \Omega\} \end{aligned}$$

and let Π^{PF} be the Pareto frontier of the projection of $core(v-c)$ on the n last coordinates. Formally,

$$\begin{aligned} \Pi^{PF} &= \{(\pi_i)_{i\in N} \in \mathbb{R}^n_+ | \text{there is } \pi^b \geq 0 \text{ with } (\pi^b, (\pi_i)_{i\in N}) \in core(v-c) \\ &\quad \text{and there is no other} (\pi'^b, (\pi'_i)_{\in N}) \in core(v-c), \\ &\quad \text{such that } \pi'_i \geq \pi_i, \text{ for all } i \in N \text{ with } \pi'_j > \pi_j \text{ for at least some } j\}. \end{aligned}$$

Notice that the core of the game is defined through linear inequalities, thus it is a polytope and so is Π^{PF}. We will prove that Π^{PF} is the set of the principals' equilibrium rents. First, the following lemma states that principal i coordinate or payoff in the core is lower or equal to her marginal contribution (as in Proposition 1 in Chiesa and Denicolò (2009)), which implies that if a principal is not selling in *some* efficient bundle, then her coordinate in the core will be zero.

Lemma 1. *Let v be a value function and c a unit cost vector. If $(\pi^b, (\pi_i)_{i\in N}) \in core(v-c)$, then $\pi_i \leq mc_i$ for all $i \in N$. Therefore, $\pi_i = 0$ for all $i \notin F(\mathbf{S})$, $\mathbf{S} \in \arg\max_{\mathbf{T}} (v-c)(\mathbf{T})$.*

Proof: Given $i \in N$, let $\mathbf{S} \in \arg\max_{\mathbf{T}\subseteq\Omega, T_i=\emptyset} : (v-c)(\mathbf{T})$, so that $v(\mathbf{S}) = V^*_{-i}$. If $(\pi^b, (\pi_i)_{i\in N}) \in core(v-c)$, then by definition $V^* = \pi^b + \sum_{j\in N} \pi_j$ and $V^*_{-i} \leq \pi^b + \sum_{j\in F(\mathbf{S})} \pi_j$. Because $\pi_j \geq 0$ for all $j \in N$, then $V^*_{-i} \leq \pi^b + \sum_{j\in(N\setminus i)} \pi_j$. Thus $V^* - V^*_{-i} = mc_i \geq \pi_i$.

Now, if $i \notin F(\mathbf{S})$ for some $\mathbf{S} \in \arg\max_{\mathbf{T}} : (v-c)(\mathbf{T})$, then $V^* = V^*_{-i}$ and $mc_i = \pi_i = 0$. ■

The following result states that there is a bijection between the core and the set of price vectors: first, each element $(\pi_i)_{i\in N} \in \Pi^{PF}$ has associated an equilibrium price vector (a contract vector) such that π_i is firm i's profit or rent; second, if $\mathbf{p}$ is an *SPSE*-price vector, then the corresponding firms' profit vector belongs to Π^{PF}. In the Appendix it is proven,

Proposition 4. *For every value function v and unit cost vector c it is verified that*

i) if $(\pi_i)_{i\in N} \in \Pi^{PF}$, then $(\mathbf{S}, \mathbf{p}) \in SPSE$, where $\mathbf{S}$ is any socially efficient consumption set and $p_i(T_i) = \pi_i + c_i(T_i)$ for all $i \in N, T_i \subseteq \Omega_i$,

ii) if $(\mathbf{S}, \mathbf{p})$ is an SPSE*-outcome, then $(\pi_i)_{i\in N} \in \Pi^{PF}$, where $\pi_i = (p_i - c_i)(S_i)$, $i \in N$.*

The above Proposition 4 and Lemma 1 characterize the principals' equilibrium payoffs. First, the principals' equilibrium payoffs are not unique and are characterized by the polytope formed by their most preferred points in the core of the agency game G^{MB}. Second, the maximum payoff of a firm is bounded from above by its marginal contribution. Third, a firm would only obtain a positive profit if it were selling at least a component of every efficient bundle. Fourth, if none of the firms sells at least one component of every efficient bundle, then every equilibrium consumption set is offered at unit cost prices, firms' payoffs are zero and the consumer extracts the entire social surplus. This is the case when principals are identical (they produce the same substitute products with the same technology as in the classical Bertrand Theory), or there are two or more socially efficient exclusive dealing bundles.

The intuition is clear, suppose two firms and the market for systems of our previous examples. If no principal sells at least one component of every efficient bundle, then there exist at least two efficient systems: the exclusive dealing bundles. By individual excludability (condition FC1) at equilibrium $(v - c)(a, b) = (v - c)(c, d)$. By Lemma 1, firms' profits are zero and then by Proposition 4, $p_{ab} = v(a, b) - \max\{(v - c)(c, d), 0\} = c_a + c_b$, $p_{cd} = v(c, d) - \max\{(v - c)(a, b), 0\} = c_c + c_d$ and therefore $\pi_1 = \pi_2 = 0$ and the agent obtains all the surplus. Notice that $mc_1 = mc_2 = 0$. Next, suppose that only one firm, say Firm 1, sells at least one component of every efficient bundle. This can be only if $\{a, b\}$ is the efficient bundle but $\{c, d\}$ is not. Then, again by condition FC1, the profit of Firm 1 is $\pi_1 = (v-c)(a, b)-\max\{(v-c)(c, d), 0\}$, which will be positive as long as $(v-c)(a, b) > (v-c)(c, d)$, $\pi_2 = 0$ and the agent obtains $(v - c)(c, d)$. Notice that $mc_1 = (v - c)(a, b) - (v - c)(c, d) = \pi_1$ and $mc_2 = 0 = \pi_2$. Therefore, if in all socially efficient consumption sets the agent chooses only products of the same firm (exclusive dealing), say i, then that firm will obtain its marginal contribution, i.e., $\pi_i = mc_i$, the products of any other firm will be offered at unit cost prices, and the agent will obtain a positive payoff equal to V^*_{-i}. The same results will be obtained if there is an exclusive dealing equilibrium, with products belonging to say firm i, as well as another or several (partial) common agency equilibria. Then, all firms but i will obtain zero profits and firm i's payoff will be its marginal contribution.

On the contrary, when the equilibrium consumption set is a common agency bundle (i.e., it contains products of two or more principals), then although the principals might offer their products as bundles at special prices, the agent selects a subset of products of each firm. For example, suppose that both Firm 1 and 2 sell at least one component of every efficient system. The efficient bundles are either $\{a, d\}$ or $\{b, c\}$, or $\{a, d\}$ and $\{b, c\}$. In each of them, by strong stability (condition FC4) p_{ab} and p_{cd} are higher than unit costs at equilibrium with either $p_{ab} = p_a$, $p_{cd} = p_d$ or $p_{ab} = p_b$, $p_{cd} = p_c$ or both, and principals' profits are positive. By Lemma 1, it is interesting to notice that if the sum of all principals' marginal contributions is lower than or equal to the maximum social surplus V^*, then all equilibrium outcomes will be payoff equivalent and in each equilibrium the principals will obtain their marginal contribution and the agent a positive surplus. This is the case in Example 1, where $V^* = 9$ and the marginal contributions are $mc_1 = 1$ and $mc_2 = 5$. In the unique $SPSE$ these values are firms' payoffs and the consumer surplus is 3. On the contrary, if the sum of all principals' marginal contributions is greater than V^*, then there will be different price vector equilibria but in all of them the consumer surplus will be zero. In Example 2, $V^* = 9$, $mc_1 = 9 - \delta$ and $mc_2 = 5$, with $0 < \delta < 1$. Recall that the efficient bundle was $\mathbf{S} = \{a, d\}$ and there were infinite equilibria depending on the price vector

$$\begin{aligned} p_a + p_d &= 9,\ 4 \leq p_a \leq 9 - \delta,\ \delta \leq p_d \leq 5, \text{ and} \\ p_k &= M, k \in \{b, c, \{a, b\}, \{c, d\}\}. \end{aligned}$$

Thus, in all efficient equilibria the joint principals' payoffs are 9, so that the consumer surplus is zero. However, depending on the equilibrium price vector, the principals' payoffs will be different. Namely, if principal 1's rent is his marginal contribution $9 - \delta$ (his most preferred point in the core of the game), then principal 2' rent will be $\delta < mc_2 = 5$, and viceversa[12].

The Pareto dominant equilibrium for the principals is denoted as the *minimum rent* equilibrium by Chiesa and Denicolò (2009), since it minimizes the agent's payoff. Notice, however that when the sum of principals marginal contributions exceeds the maximum social surplus, the consumer's surplus is always zero and each principal's marginal contribution is his more preferred point in the core, then by Lemma 1 only one principal can get his preferred rent. By the above results we can conclude:

Corollary 2. *1) The minimum rent equilibria of G^{MB} when the sum of all principals' marginal contributions is lower than or equal to the maximum social surplus is outcome equivalent to the truthful equilibrium. 2) When the sum of all principals' marginal contributions is greater than the maximum social surplus, then there will be infinite equilibrium price vectors but in all of them the consumer surplus will be zero; only one principal will obtain its marginal contribution in his preferred minimum rent equilibria but the others will obtain less than their marginal contributions.*

The results in Collorary 2 differs from those of the delegated agency literature and, in particular, from those of Chiesa and Denicolò (2009). They show that if the number of principals is higher than 2, then at the minimum rent equilibria every principal's payoff exceed his marginal contribution to the social surplus, where the existence of strictly increasing cost explains these authors' results. In our case, given the heterogeneity of firms' products, when the efficient consumption set is a mixed bundle and the other bundles are not too valued by the agent, the marginal contribution of each principal is very high and the sum of all of them may exceed the value of the social surplus. Since the agent's rent is zero and all the marginal contributions are equal, then by Lemma 1, only one principal can get its preferred

[12] It is easy to show that the same results obtain with positive (constant) costs of production.

rent. Therefore, the product heterogeneity and the lack of monotonicity of the agent's value function for bundles drive the above results.

The precise relationship between the maximum social surplus and the sum of the principals' marginal contributions is difficult to obtain in general, unless we know some properties of the value function. Next, we explore a question that deserves some attention.

4.3. Structure of the principals' equilibrium strategies: number of out of equilibrium offers

We have seen that when the equilibrium consumption set is a common agency bundle, mixed bundling contracts are out-of-equilibrium offers sustaining the equilibrium consumption sets of individual components. These contracts are the reason of the multiplicity of equilibrium price-vectors. In view of the role that the out of equilibrium offers play sustaining the equilibrium schedules, an interesting question to discuss is the minimum number of them needed to support *SPSE*-outcomes.

We can distinguish two cases. When the equilibrium outcome is *exclusive dealing*, condition FC1, or *individual excludability*, will suffice to price the equilibrium bundle and *strong stability* or condition FC4 will guarantee the equilibrium condition for any subset of agents (no deviations). Thus, only the equilibrium offer and the null offer (TIOLI) offers are needed to support exclusive dealing equilibrium outcomes.

However, when the equilibrium outcome is *delegated common agency*, where several firms are active in the equilibrium consumption set, *individual excludability* and *strong stability* will require at least three offers. At equilibrium, each active principal could need to price its equilibrium bundle, at least another of its own bundles, and the null bundle. The pricing of out of equilibrium bundles by each principal avoids his deviation to exclusive dealing outcomes. Therefore out of equilibrium offers by each principal are needed to sustain equilibrium outcomes. For instance, the delegated common agency equilibria in the market for systems of Example 1 implies that the equilibrium consumption set is $\{a, d\}$. To sustain the equilibrium each principal needs to offer three bundle prices. Namely, $p_{ab} = p_a$, $p_{cd} = p_d$ and the null offer. In general, the out of equilibrium bundles priced by each principal will depend on the agent's preferences as shown in the following example.

Example 3: Let the set of firms (principals) be $N = \{1, 2, \dots, n\}$, each one producing two products $\Omega_i = \{a_i, b_i\}$ for all $i \in N$. Assume for simplicity that production costs are zero. Let $A = \{a_1, \dots, a_n\}$ and for all $i \in N$ let $B_{-i} = \{b_1, \dots, b_{i-1}, b_{i+1}, \dots, b_n\}$. In bundle A all firms sell their first product, meanwhile in the other bundles all firms but one sell their second product. The agent's value function is $v(A) = \alpha$ and $v(B_{-i}) = \beta$ where $\alpha > \beta > 0$ and $v(S) = 0$ otherwise. Thus, the agent's most valuable bundle is A; she values less the bundles B_{-i}, and any other bundle is worthless for her.

The efficient consumption set is the common agency bundle A, the equilibrium prices are $p_{a_i} = \alpha - \beta$, and the out-of-equilibrium prices are $p_{b_i} = \alpha - \beta$, for all $i \in N$. The consumer surplus is $v(A) - \sum_i p_{a_i} = \alpha - n(\alpha - \beta)$, therefore we assume that $\beta > \frac{n-1}{n}\alpha$ to guarantee a positive agent's surplus. Since there are not costs, the marginal contribution of principal i is given by $v(A) - v(B_{-i}) = \alpha - \beta$, which is equal to his payoff. In this simple example each principal needs to price his two products, the sold one, the unsold one and the null offer.

Notice that if each principal i priced only a_i as $p_{a_i} = \alpha - \beta$, and set b_i equal to marginal cost (which is zero), the agent would deviate to any of bundles B_{-i}, since $v(A) - \sum_i p_{a_i} = \alpha - n(\alpha - \beta) < \beta = v(B_{-i})$, for any i.

Chiesa and Denicolò (2009) found (see Proposition 3, pag. 305) that in their model any equilibrium outcome can be supported by equilibrium supply schedules (pairs of product-price) such that two principals offer two supply schedules (the empty set and one bundle), and the remaining $n - 2$ principals offer three supply schedules (the empty set and two different bundles). In our *delegated common agency equilibrium, all principals may need to offer at least three supply schedules.* The reason behind this result is the deterrence of each principal's deviations to his exclusive dealing outcomes. Given the non-monotonicity of the agent's value function for bundles, the precise number of supply schedules will depend on the particular details of the model.

5. Monotonic social surplus functions: Delegated Common Agency.

In this section we study the surplus sharing between the agent and the principals in delegated common agency, when the social surplus function is monotonic. In this context, exclusive dealing is never an equilibrium consumption set, and the agent will contract with either a subset of principals (partial delegated common agency) or with all of them (delgated common agency). We show that when the social surplus function is supermodular, then the agent's surplus is zero and the set of principals' rents is completely characterized by the convex hull of the vertex arising from their accumulative marginal contributions, which coincides with the Pareto frontier of $core(v-c)$. We also prove that when the principals are substitutes (a more general condition than strong subadditivity), then at any equilibrium the principals' rents are their marginal contributions (truthfull equilibrium) and the agent obtains the difference between the social marginal contribution of her consumption set Ω and the sum of the social marginal contributions of the principals $\sum mc_i$. In both cases of monotonic social surplus functions TIOLI offers are able to sustain the *SPSE* outcomes.

Let us introduce some convention in notations. Let $\Omega \backslash \mathbf{S}$ be $(\Omega_1 \backslash S_1, \ldots, \Omega_n \backslash S_n)$ and let $v(S_i)$ be $v(\emptyset, \ldots, S_i, \ldots \emptyset)$ i.e., a consumption set where the agent only buys from principal i. Given $w \in \Omega_i$ and $\mathbf{S} \subseteq \Omega$, let $\mathbf{S} + w$ be $(S_1, \ldots, S_i + \{w\}, \ldots, S_n)$, with $S_i + \{w\} = S_i \cup \{w\}$.

Definition 3. *i)* $(v-c)$ *is* monotonic *if and only if* $(v-c)(\mathbf{S}) \leq (v-c)(\mathbf{T})$ *whenever* $\mathbf{S} \subseteq \mathbf{T} \subseteq \Omega$,

ii) $(v-c)$ *is* submodular *if and only if* $(v-c)(\mathbf{T}+w) - (v-c)(\mathbf{T}) \leq (v-c)(\mathbf{S}+w) - (v-c)(\mathbf{S})$ *whenever* $\mathbf{S} \subseteq \mathbf{T} \subseteq \Omega \backslash w$, *and*

iii) $(v-c)$ *is* supermodular *if the opposite inequality holds.*

Thus, the monotonicity of $(v-c)$ implies that the social surplus increases for larger consumption sets. If $(v-c)$ is monotonic, then, by Theorem 1, there is always an *SPSE*-outcome with Ω as the equilibrium consumption set. Furthermore, if $(v-c)$ is strictly monotonic, then Ω is the unique equilibrium consumption set. Thus under strict monotonicity of the social value function, G^{MB} is a delegated common agency game where the agent contracts with all the principals (common agency). Moreover, by Lemma 1 and Proposition 4 we know that,

Lemma 2. *If* $(\Omega, \mathbf{p}) \in SPSE$*-outcome set, then* $p_i(\Omega_i) - c(\Omega_i) \leq mc_i = (v-c)(\Omega) - (v-c)(\Omega \backslash \Omega_i)$ *for all* $i \in N$.

Suppose now that the social surplus function is *submodular*. This is a condition that seems particularly attractive since it expresses, in the case of indivisible goods, the idea that the marginal utility of an item decreases when the bundle of goods to which it is added gets larger. The submodularity of $(v-c)$ reflects a kind of substitution among products or bundles of products so that there is competition among principals and the agent will obtain some surplus.

Let us extend the concept of principal's marginal contribution to a set of principals: the *marginal contribution of a set of principals* $A \subseteq N$, say mc_A, is the difference between the maximum social surplus attainable V^* and the maximum social surplus attainable when set A of principals is inactive V^*_{-A}, i.e., $mc_A = V^* - V^*_{-A}$, where $V^*_{-A} = \max_{\substack{\mathbf{S} \subseteq \Omega \\ S_i = \emptyset, i \in A}} : (v-c)(\mathbf{S})$.

Following Shapley (1962), we say that principals are *substitutes* if the social marginal contribution of set A is bigger than or equal to the sum of the marginal contributions of firms in A,

$$mc_A \geq \sum_{i \in A} mc_i \qquad \forall A \subseteq N. \tag{FS}$$

The next proposition states that the equilibrium prices of monotonic social surplus functions satisfying FS are equal to the social marginal contributions of the principals plus their corresponding marginal costs. In the Appendix it is proven.

Proposition 5. *Let* $(v-c)$ *be a monotonic social surplus function and suppose that principals are substitutes (FS holds). Then* $(\mathbf{S}^*, \mathbf{p}^*) \in SPSE$*-outcome set, with* $(v-c)(\mathbf{S}^*) = (v-c)(\Omega)$ *and* $p^*_i(T_i) = mc_i + c(T_i)$*, for all* $T_i \subseteq \Omega_i$*, and* $i \in N$*. The converse is also true.*
If $(v-c)$ *is strictly monotonic, then the unique equilibrium consumption set is* Ω.

Therefore, each principal i sells S^*_i as a bundle and obtains its marginal contribution as its profits, $(p^*_i - c_i)(S^*_i) = mc_i$. Moreover, the agent surplus is positive, reflecting market competition under FS:

$$\begin{aligned} cs(S^*) &= v(S^*) - \sum_{i \in F(S^*)} p^*_i(S^*_i) = (v-c)(S^*) - \sum_{i \in F(S^*)} (p^*_i - c_i)(S^*_i) \\ &= mc_{F(\mathbf{S}^*)} - \sum_{i \in F(\mathbf{S}^*)} mc_i \geq 0. \end{aligned}$$

Two straightforward results are the following: 1) if $(v-c)$ is a submodular social surplus function, then principals are substitutes, i.e. FS is satisfied; thus the FS condition is more general than the concavity (or strong subadditivity) condition in the literature; and 2) if

$(v-c)$ is a submodular value function and $mc_i \geq 0$ for all $i \in N$, then $(v-c)$ is monotonic. Then, trivially from Proposition 5 we obtain the next result.

Corollary 3. *Let $(v-c)$ be submodular and $mc_i \geq 0$ for all $i \in N$. Thus, principals are substitutes, and for all $i \in N$, principal i's equilibrium rent in any* SPSE-*outcome is equal to his social marginal contribution.*

By the above Proposition (5) principals are substitutes, and for all $i \in N$, principal i's price is $p_i^*(\Omega_i) = v(\Omega) - v(\Omega \backslash \Omega_i) + c(\Omega_i)$ and his rent is his social marginal contribution. The agent is indifferent between buying Ω or $\Omega \backslash \Omega_i$. Therefore, each principal's strategy of selling his set of products as an indivisible bundle is an equilibrium outcome and there is not need to set prices out of the equilibrium offer. Thus, TIOLI is an efficient equilibrium outcome.

Corollary 4. *Let $(v-c)$ be submodular and $mc_i \geq 0$ for all $i \in N$. Then, the minimum rent equilibrium is outcome equivalent to the truthfull equilibrium. Moreover only the equilibrium offer and the null offer (TIOLI) are needed to sustain the* SPSE *outcome.*

Now suppose that the social surplus function is *supermodular*. The supermodularity of $(v-c)$ reflects complementarities among products or bundles of products and hence among principals. Therefore, it induces only weak market competition so that principals can extract the entire agent surplus. The higher degree of complementary among principals translates into a higher marginal contribution of each of them, so that their sum is bigger than the social surplus (and property FS does not hold). In this framework, principals extract all the social surplus, but not all of them can obtain at equilibrium their social marginal contributions.

As the next proposition states, the set of principals' equilibrium rents is a polytope, whose corners are the vectors of the principals' equilibrium rents verifying that at least one principal obtains his social marginal contribution. The intuition is as follows. Assume that principals are ordered, so that the first principal is principal 1, the second principal is principal 2 and so on. A market with only a consumer has a null social surplus; if principal 1 enters in that market, then the social surplus will increase in $V^*_{-(N\backslash 1)} - V^*_{-N}$; if now principal 2 enters, then the social surplus will increase in $V^*_{-(N\backslash\{1,2\})} - V^*_{-N\backslash 1}$; in general, after the entry in the market of principal i, the social surplus will increase in $V^*_{-(N\backslash\{1,\ldots,i\})} - V^*_{-(N\backslash\{1,\ldots,i-1\})}$; the entry of the last principal, principal n, increases the social surplus in $V^* - V^*_{-n}$.

Notice that in the above process only principal n generates an increase in the social surplus equal to his marginal contribution, and that the sum of all these increases is equal to V^*, the social surplus. The next proposition states that the vector of the social surplus increases is a corner in the polytope of the set of principals' equilibrium rents. Thus, the different orders in the set of principal generates the different corners, and the convex hull of these corners are also feasible equilibrium rents.

Formally, let Σ be the set of permutations (orderings) of principals (permutation of N) and let $\sigma \in \Sigma$ be any of its elements. Let P_i^σ be the set of principals which precede principal i with respect to permutation σ, i.e., for all $i \in N$ and $\sigma \in \Sigma$, $P_i^\sigma = \{j \in N | \sigma(j) < \sigma(i)\}$.

Define, following Shapley (1971), the marginal contribution vector $x^\sigma(v-c) \in \mathbb{R}^n$ of $(v-c)$ with respect to ordering σ by,

$$x_i^\sigma(v-c) = V^*_{-N\setminus(P_i^\sigma+i)} - V^*_{-N\setminus P_i^\sigma}, \qquad \text{for all } i \in N,$$

so that, $x_i^\sigma(v-c)$ is the principal i's social contribution when he enters in a market where only principals in P_i^σ were active (the ones preceding him with respect to ordering σ).

Moreover, it is straightforward to prove that if $(v-c)$ is nonnegative and supermodular, then $(v-c)$ is monotonic. In the Appendix it is proven,

Proposition 6. *Let $(v-c)$ be a supermodular value function, such that $mc_i \geq 0$ for all $i \in N$. Then the agent's surplus is zero and the principals' equilibrium profits in any* SPSE-*outcome are $conv\{x^\sigma(v-c)|\sigma \in \Sigma\}$.*

By Proposition 6, the principals' equilibrium rents are characterized by set $conv\{x^\sigma(v-c)|\sigma \in \Sigma\}$, which turns out to be the Pareto frontier of $core(v-c)$. Thus, when $(v-c)$ is supermodular, then the core of the value function is always priced by subgame perfect strong equilibrium payoffs.

Under monotonic and supermodular social surplus functions, again TIOLI offers sustain the equilibrium outcome. The reason is that the consumer surplus is zero and thus she is indifferent between buying all products or buying none. Hence, each principal, say i, only has to price his own bundle Ω_i and the null offer.

In view of the Propositions of this section, we can assert that the lack of monotonicity of the agent's preferences over bundles of goods and hence the inherited lack of monotonity of the social value function is what drives our results on the number of out of equilibrium offers and on the principals' rents in our general model.

6. Concluding Remarks

This paper has contributed to the literature on delegated common agency and complete information by extending the insights by Berheim and Whinston (1986, a,b) to multi-product markets with indivisibilities and no necessarily monotone agent's value function for bundles.

First, we have characterized the equilibrium outcomes in these settings and, by considering a kind of extended contracts -mixed bundling contracts- that stresses the role of out-of-equilibrium offers, we have shown the equilibrium existence. Under mixed bundling contracts the agent has the option of buying bundles of goods from a firm at a discount over the single good prices. In our model, with multi-product firms and an agent with preferences over each bundle of goods, mixed bundling contracts are conditional on exclusive dealing for each bundle of two or more goods. Therefore, these contracts can be seen as either an aggressive pricing policy for exclusive dealing outcomes or as out-of-equilibrium offers sustaining the equilibrium consumption sets of individual components in delegated common agency allocations. The discrimination on exclusivity both facilitates collusion on common agency outcomes and represents a credible threat that avoids deviations by the principals, thus helping them set incentive-compatible contracts. We have also found that equilibrium need not be unique in the sense that many equilibrium price vectors may sustain the same equilibrium allocation. This is due to the fact that each principal offers contracts for its products and also offers subsets of them as bundles at a special price. Notice that this implies that we have not considered singleton contracts (direct mechanisms) in the delegated agency game. Such mechanisms do not allow for any offer to remain unchosen in equilibrium; in other words, out-of-equilibrium messages are possible and they have a commitment value. Furthermore, efficient and inefficient equilibria may belong to the sub-game Nash correspondence. The lack of coordination among the principals is the reason behind inefficient equilibria, in which the agent chooses a suboptimal bundle and no principal has a profitable deviation inducing the agent to buy the surplus-maximizing bundle.

Second, we have ruled out inefficient equilibria by either assuming that all firms are pricing unsold bundles at the same profit margin as the bundle sold at equilibrium, or imposing the solution concept of subgame perfect strong equilibrium, which requires the absence of profitable deviations by any subset of principals and the agent. Given the lack of monotonicity of our general model, our rationale for equilibrium menus which take into account deviations by any set of firms and the player is less demanding than that of coalition-proof equilibrium which requires the equilibrium immunity to deviations by subsets of principals which are themselves immune to deviations by sub-coalitions, etc.

Third, we have shown that when the equilibrium consumption set is a common agency bundle, mixed bundling contracts are out-of-equilibrium offers sustaining the equilibrium consumption sets of individual components. Then we have analyzed the *minimum number* of them needed to support *SPSE*-outcomes.In our delegated common agency equilibrium, all principals may need to offer at least three supply schedules. The reason behind this results was the deterrence of each principal's deviations to his exclusive dealing outcomes.

Finally, we have analyzed the specific structure of equilibrium prices and payoffs for common agency outcomes when the social surplus function is monotone and either submodular or supermodular. In the former case, principals are substitutes and their equilibrium rents are equal to the principals' social marginal contributions with an agent's positive rent, thus reflecting market competition. In the latter case, the agent's rent is zero and then the core of the value function is always priced by the subgame perfect Nash-equilibrium rents. These results allow us conclude that the lack of monotonicity of the agent's preferences over bundles of goods and hence the inherited lack of monotonity of the social value function drives our results on the number of out of equilibrium offers and the principals' rents of the general model.

An interesting extension of our analysis is to consider both multiple principals and agents. Our intuition is that a new kind of extended contracts, *non-anonymous* and mixed bundling contracts, is needed to ensure the equilibrium existence. This is left for future research.

7. Appendix.

Non-existence of equilibrium linear prices in Example 1 *(The market for systems)***.** The efficient consumption set is the common agency bundle $\mathbf{S} = \{a, d\}$. Suppose that

$\delta = 8$. Let us show that there are not linear prices (contracts) guaranteeing the existence of any equilibrium outcome. Traditional arguments suggest how to set prices for, say, the efficient consumption set $\mathbf{S} = \{a, d\}$ (the proof of the remaining cases is similar). Whether the agent's participation constraint is binding will help define the equilibrium prices. Suppose that the agent's participation constraint is not binding, i.e. $cs[\mathbf{S}, \mathbf{p}] = 9 - p_a - p_d > 0$, or $p_a + p_d < 9$. Since $\mathbf{S}$ is a two-firm bundle, then firms' prices have to first guarantee the incentive compatibility constraints avoiding exclusive dealing, i.e., the agent's surplus associated to the consumption of set $\mathbf{S} = \{a, d\}$, has to be bigger than or equal to the agent's surplus if either firm 2 or firm 1 were removed from the market,

$$cs[\mathbf{S}, \mathbf{p}] = 9 - p_a - p_d \geq 4 - p_a - p_b = cs[\{a, b\}, \mathbf{p}], \tag{7}$$
$$cs[\mathbf{S}, \mathbf{p}] = 9 - p_a - p_d \geq 8 - p_c - p_d = cs[\{c, d\}, \mathbf{p}]. \tag{8}$$

Notice, however, that both (7) and (8) have to be binding since otherwise either firm 1 could deviate by raising p_a or firm 2 by increasing p_d and be better off, without changing the agent's choice. Thus, we have that $p_d = 5 + p_b$ and $p_a = 1 + p_c$ and hence that $p_a + p_d = 6 + p_b + p_c$.

The multiproduct nature of the model includes another incentive constraint dealing with the agent's switching to the other common agency bundle, i.e., $cs[\mathbf{S}, \mathbf{p}] = 9 - p_a - p_d \geq 5 - p_b - p_c = cs[\{b, c\}, \mathbf{p}]$, which implies that $p_a + p_d \leq 4 + p_b + p_c$ that contradicts the previous result. Hence the agent's participation constraint has to be binding and thus $p_a + p_d = 9$.

Since, $p_a + p_d = 9$ the agent's surplus is zero and firms 1 and 2's profits are $9 - p_d$ and $9 - p_a$, respectively. However, firm 1 could deviate by setting $p_b = 0$ and $p_a = 4$ and trying to sell its own bundle $\{a, b\}$. Therefore, at equilibrium it must be that $9 - p_d \geq 4$ or $p_d \leq 5$; similarly, to avoid firm 2' deviation by setting $p_c = 0$ and $p_d = 8$, and selling its own bundle $\{c, d\}$, it must be that $p_a \leq 1$. Hence $p_a + p_d \leq 6$ which contradicts the assumption that $p_a + p_d = 9$.

Therefore, no linear prices support $\mathbf{S} = \{a, d\}$ as the equilibrium consumption set. The above reasoning can be applied to any other bundle to conclude that *under linear pricing the equilibrium may fail to exist.*

Equilibrium mixed-bundling prices in Example 1 (continuation). The efficient bundle $\mathbf{S} = \{a, d\}$ is the unique *SPE*-consumption set, supported by equilibrium prices:

$$p_a = p_{ab} = v(a, d) - v(c, d) = 1,\ p_d = p_{cd} = v(a, d) - v(a, b) = 5,\ p_b + p_c \geq 2,$$

and the consumer surplus is $cs(\{a, d\}, p) = 9 - 6 = 3 > 0$.

The following steps prove that the above outcome is the unique equilibrium:

1) Suppose that the agent's participation constraint is binding: $p_a + p_b = 9$. Since

$$cs(\mathbf{S}, p) = 9 - p_a - p_d \geq 4 - p_{ab} = cs(\{a, b\}, p),$$
$$cs(\mathbf{S}, p) = 9 - p_a - p_d \geq 8 - p_{cd} = cs(\{c, d\}, p),$$

then $p_{ab} \geq 4$ and $p_{cd} \geq 8$. If $p_a < 4$, then firm 1 is better off first setting p_a and p_b high enough and then setting $p'_{ab} = 4 - \varepsilon > p_a$, for ε sufficiently small and then the agent will switch to firm 1's exclusive dealing bundle $\{a, b\}$ obtaining a profit equal to ε. Then, $p_a \geq 4$. We can reproduce the same argument for $p_d < 8$ to obtain that $p_d \geq 8$. But this contradicts out statement that $p_a + p_b = 9$ and therefore the agent's participation constraint is not binding.

2) Let us see that the incentive compatibility constraints avoiding exclusive dealing have to be binding,

$$cs(\mathbf{S}, p) = 9 - p_a - p_d = 4 - p_{ab} = cs(\{a, b\}, p), \tag{9}$$
$$cs(\mathbf{S}, p) = 9 - p_a - p_d = 8 - p_{cd} = cs(\{c, d\}, p), \tag{10}$$

because if the left-hand side of either 9 or 10 were strictly bigger than its corresponding right-hand side, then some firm would have an incentive to profitably raise its price. More precisely, notice that if both 9 and 10 were satisfied with strict inequality, then both firms will have an incentive to rise the prices of p_a and p_d. On the other hans, if only 9 were satisfied with strict inequality, then firm 2 will have an incentive to rise p_d until inequality 9 becomes in an equality, rising simultaneously p_{cd} to maintain 10 as an equality. Similar argument could be applied if only 10 were satisfied with strict inequality.

3) Suppose now that $p_{ab} > p_a$. Then, by equation 9 firm 1 could be better off by setting p_a and p_b high enough and then decreasing the price of $\{a, b\}$ to $p_{ab} - \varepsilon > p_a$ and making the agent switch from $\{a, d\}$ to $\{a, b\}$. The same reasoning applies to $p_{cd} > p_d$. Therefore, at equilibrium $p_a \geq p_{ab}$ and $p_d \geq p_{cd}$.

4) Consider then that $p_a \geq p_{ab}$ and $p_d \geq p_{cd}$, by equations 9 and 10,

$$p_a + p_d = 5 + p_{ab} \text{ or } p_d = 5 + p_{ab} - p_a \leq 5,$$
$$p_a + p_d = 1 + p_{cd} \text{ or } p_a = 1 + p_{cd} - p_d \leq 1.$$

Notice, that the first equality also implies that $p_{ab} = p_a + p_d - 5$ and then $p_a + p_d \geq 5$.

5) Finally, suppose that $p_d < 4$. Then, firm 2 could be better off by setting p_c and p_d high enough and then setting the price of $\{c, d\}$ to $p_{cd} = 4 - \varepsilon > p_d$ and making the agent switch from $\{a, d\}$ to $\{c, d\}$. Thus, $p_d \geq 4$.

Therefore the an equilibrium price vector is

$$0 \leq p_a \leq 1,\ 4 \leq p_d \leq 5,\ 5 \leq p_a + p_d,$$
$$p_{ab} = p_a + p_d - 5,\ p_{cd} = p_a + p_d - 1,$$
$$p_b \text{ and } p_c \text{ bigger enough.}$$

Proof of Proposition 4: The proof of i) closely follows the steps of the proof of Theorem 1. Let $(\mathbf{S}, \mathbf{p})$ be an SPE^*-outcome, then $(\mathbf{S}, \mathbf{p}^{\mathbf{S}}) \in SPE^*$. We define $\pi_i = (p_i^S - c_i)(S_i)$, $i \in N$ and $\pi^b = (v - c)(\mathbf{S}) - \sum_{i \in N} \pi_i$. First let us prove that $(\pi^b, (\pi_i)_{i \in N}) \in core(v - c)$. By the definition of $\mathbf{p}^{\mathbf{S}}$, if $j \notin F(\mathbf{S})$, then $\pi_j = (p_j^S - c_j)(S_j) = 0$, which implies that $\pi^b = (v - c)(\mathbf{S}) - \sum_{i \in F(\mathbf{S})} \pi_i$.

Since $V^* = (v-c)(\mathbf{S})$, we have that $\pi^b + \sum_{i\in N} \pi_i = V^*$. Given $T \subseteq \Omega$ by BC

$$v(\mathbf{S}) - \sum_{i\in F(\mathbf{S})} p_i^S(S_i) \geq v(T) - \sum_{i\in F(T)} p_i^S(T_i),$$

which implies that

$$(v-c)(\mathbf{S}) - \sum_{i\in F(\mathbf{S})} (p_i^S - c)(S_i) \geq (v-c)(T) - \sum_{i\in F(T)} (p_i^S - c)(T_i)$$

and hence,

$$\pi^b + \sum_{i\in F(T)} (p_i^S - c)(S_i) \geq (v-c)(T)$$

and $(\pi^b, (\pi_i)_{i\in N}) \in core(v-c)$.

Now let us prove that $(\pi_i)_{i\in N}$ belongs to Π^{PF}. Suppose, on the contrary, that $(\pi_i)_{i\in N}$ is Pareto dominated by an element $(\pi_i') \in \Pi^{PF}$, i.e. $\pi_j' \geq \pi_j$, $j \in N$ and j_0 is such that $\pi_{j_0}' > \pi_{j_0}$. W.l.o.g. we can assume that π' is in Π^{PF}. Recall $\mathbf{S} \in \arg\max_{T\subseteq\Omega}\{(v-c)(T)$, then by Lemma 1 we have that $j_0 \in F(\mathbf{S})$, otherwise $\pi_{j_0}' = 0 > \pi_{j_o}$ which is a contradiction. Let $\mathbf{p}'$ be defined as $p_i'(T_i) = \pi_i' + c_i(T_i)$, for all $i \in N, T_i \subseteq \Omega_i$, so that $p_{j_0}'(S_{j_0}) = \pi_{j_0}' + c_{j_0}(S_{j_0}) > \pi_{j_0} + c_{j_0}(S_{j_0}) = p_{j_0}(S_{j_0})$. By i), already proven, we have that $(\mathbf{S}, \mathbf{p}')$ is an SPE^*-outcome, then firm j_0 has incentives to raise its equilibrium price vector up to $\mathbf{p\prime}$, which contradicts that $(\mathbf{S}, \mathbf{p})$ is a SPE^*-outcome. This implies that $(\pi_i)_{i\in N}$ belongs to Π^{PF}, as claimed. ■

Proof of Proposition 5: Assume w.l.o.g. that unit costs are zero. Given that v is monotonic $v(\Omega) \geq v(\mathbf{S})$ for all $\mathbf{S} \subseteq \Omega$ and $mc_i = v(\Omega) - v(\Omega\backslash\Omega_i) \geq 0$. Let $p_i^*(T_i) = p_i^*(\Omega_i) = v(\Omega) - v(\Omega\backslash\Omega_i)$. Thus, prices are all positive and $mc_i = p_i^*(\Omega_i)$.

First we prove that $(\Omega, \mathbf{p}^*) \in SPE^*$-outcome set. By the monotonicity of v, Corollary 1 and that $= p_i^*(T_i) = p_i^{*\Omega}(T_i)$ it suffices to prove that $(\Omega, \mathbf{p}^*) \in SPE$-outcome set.

Step 1: Consumer surplus is non-negative. By FS, $mc_N = v(\Omega) \geq \sum_{i\in N} mc_i = \sum v(\Omega) - v(\Omega\backslash\Omega_i) = \sum_{i\in N} p_i^*(\Omega_i)$, thus $cs = v(\Omega) - \sum_{i\in N} p_i^*(\Omega_i) \geq 0$.

Step 2: Condition BC in Proposition 1 is verified. Given $S \subseteq \Omega$, by definition, $mc_{N\backslash F(\mathbf{S})} \leq v(\Omega) - v(\mathbf{S})$ and by FS, $mc_{N\backslash F(\mathbf{S})} \geq \sum_{i\in N\backslash F(\mathbf{S})} mc_i$. Thus,

$$\begin{aligned} v(\Omega) - v(\mathbf{S}) &\geq \sum_{i\in N\backslash F(\mathbf{S})} mc_i \\ &= \sum_{i\in N\backslash F(\mathbf{S})} p_i^*(\Omega_i) = \sum_{i\in N} p_i^*(\Omega_i) - \sum_{i\in F(\mathbf{S})} p_i^*(\Omega_i) \\ &= \sum_{i\in N} p_i^*(\Omega_i) - \sum_{i\in F(\mathbf{S})} p_i^*(S_i) \end{aligned}$$

then $v(\Omega) - \sum_{i\in N} p_i^*(\Omega_i) \geq v(\mathbf{S}) - \sum_{i\in F(\mathbf{S})} p_i^*(S_i)$.

Step 3: Condition FC1 in Proposition 1 is verified. Given $j \in N$, by definition $\mathbf{p}^*(\Omega_j) = v(\Omega) - v(\Omega\backslash\Omega_j)$, then $v(\Omega) - \mathbf{p}^*(\Omega_j) = v(\Omega\backslash\Omega_j)$ which implies that $v(\Omega) - \sum_{i\in N} \mathbf{p}^*(\Omega_i) = v(\Omega\backslash\Omega_j) - \sum_{i\in N\backslash j} \mathbf{p}^*(\Omega_i)$. Thus, $S^j = \Omega\backslash\Omega_j$ is the one which verifies FC1 for all $j \in N$.

Step 4: Condition FC2 in Proposition 1 is verified. This condition holds trivially since $\mathbf{p}^*(S_i) = \mathbf{p}^*(\Omega_i)$ for all $i \in N, S_i \subseteq \Omega_i$.

Condition FC3 in Proposition 1 does not apply given that $N = F(\Omega)$.

Thus $(\Omega, \mathbf{p}^*) \in SPE^*$-outcome set.

Now, let us prove the converse. Consider $(\mathbf{S}^*, \mathbf{p}^*) \in SPE$-outcome set, so that $v(\mathbf{S}^*) = v(\Omega)$ and $mc_{N\backslash F(\mathbf{S}^*)} = 0$. Then by FS, for all $T_i \subseteq \Omega_i$ and $i \in N \setminus F(S^*)$

$$0 = mc_{N\backslash F(\mathbf{S}^*)} \geq \sum_{i\in N\backslash F(\mathbf{S}^*)} mc_i \geq \sum_{i\in N\backslash F(\mathbf{S}^*)} p_i^*(T_i) \geq 0,$$

which implies that $p_i^*(T_i) = 0$.

Moreover, by BC $v(\Omega) - \sum_{i\in N} p_i^*(\Omega_i) \geq v(\mathbf{S}^*) - \sum_{i\in F(\mathbf{S}^*)} p_i^*(S_i^*)$. Therefore,

$$0 = v(\Omega) - v(\mathbf{S}^*) \geq \sum_{i\in F(\mathbf{S}^*)} p_i^*(\Omega_i) - p_i^*(S_i^*).$$

Thus, $p_i^*(\Omega_i) = p_i^*(S_i^*)$ for all $i \in F(\mathbf{S}^*)$.

Proof of Proposition 6: Let Σ be the set of permutations (orderings) of $N = \{1, 2, ..., n\}$ and let $\sigma \in \Sigma$ be any of its elements. Let P_i^σ be the set of firms which precede firm i with respect to permutation σ, i.e. for all $i \in N$ and $\sigma \in \Sigma$, $P_i^\sigma = \{j \in N | \sigma(j) < \sigma(i)\}$.

Define, following Shapley (1971), the marginal contribution vector $x^\sigma(v - c) \in R^n$ of $(v - c)$ with respect to ordering σ by, $x_i^\sigma(v - c) = V(P_i^\sigma + i) - V(P_i^\sigma)$, for all $i \in N$. If $(v - c)$ is super modular, then the marginal contribution vector $x^\sigma(v - c)$ will be positive. The equality between the sets $conv\{x^\sigma(v - c)|\sigma \in \Sigma\}$ and $core(v - c)$ is given in Driessen (1993), but this equality implies that $\pi^b = 0$ and hence that $core(v - c) = \Pi^{PF}$. Thus, by Proposition 4 consumer surplus is zero and the equilibrium prices are the $core(v - c)$. ■

References

W. J. Adams and J. l. Yellen, Commodity Bundling and the Burden of Monopoly, The Quarterly Journal of Economics, 90 (3) (1976) 475-498.

I. Arribas and A. Urbano, Nash Equilibrium in a Model of Multiproduct Price Competition: an assignment Problem, Journal of Mathematical Economics, 41 (3) (2005) 351-385.

M. Armstrong and J. Vickers, Competitive Price Discrimination, The RAND Journal of Economics, 32 (4) (2001) 579-605.

R. Aumann, Acceptable points in general cooperative n-person games, Contribution to the Theory of Games IV, Annals of Mathematics Study, Princeton University Press, 40 (1959) 287-324.

B. D. Bernheim and M. D. Whinston, Common Agency, Econometrica, 54 (4) (1986) 923-942.

B. D. Bernheim and M. D. Whinston, Menu Auctions, Resource Allocations and Economic Influence, The Quartely Journal of Economics, 101 (1) (1986) 1-31.

J. Carbajo, D. de Meza, and D. J. Seidmann, A Strategic Motivation for Commodity Bundling, Journal of Industrial Economics, 38 (3) (1990) 283-298.

G. Carmona and J. Fajardo, Existence of equilibrium in common agency games with adverse selection, Games and Economic Behavior, 66 (2) (2009) 749-760.

G. Chiesa and V. Denicolò, Trading with a Common Agent under Complete Information: A Characterization of Nash Equilibria, Journal of Economic Theory, 144 (1) (2009) 296-311.

J. P. Choi, Preemptive R&D, Rent Dissipation, and the "Leverage Theory", The Quarterly Journal of Economics, 111 (4) (1996) 1153-1181.

J. P. Choi, Antitrust Analysis of Tying Arrangements, CESifo Working Paper Series 1336, CESifo Group Munich,(2004).

T. S. H. Driessen, Generalized Concavity in Game Theory: Characterizations in Terms of the Core, Faculty of Applied Mathematics. University of Twente (1993).

M. Ivaldi and D. Martimort, Competition under nonlinear Pricing, Annales d'Economie et de Statistique, 34 (1994) 71-114.

D. Laussel and M. Le Breton, Conflict and Cooperation: The Structure of Equilibrium Payoffs in Common Agency, Journal of Economic Theory, 100 (1) (2001) 93-128.

C. H. Liao and Y. Tauman, The role of Bundling in Price Competition, International Journal of Industrial Organization, 20 (3) (2002) 365-389.

C. H. Liao and A. Urbano, Pure Component Pricing in a Duopoly, The Manchester School, 70 (1) (2002) 150-163.

D. Martimort, Multi-contracting Mechanism Design, Advances in Economics and Econometrics: Theory and Applications, series "Econometric Society Monographs", Cambridge University Press (2007).

D. Martimort and L. A. Stole, Contractual Externalities and Common Agency Equilibria, The B. E. Journal of Theoretical Economics, De Gruyter, 3 (1) (2003) 1-40.

D. Martimort and L. A. Stole, Market Participation under Delegated and Intrinsic Common-Agency Games, The RAND Journal of Economics, 40 (1) (2009) 78-102.

R. P. McAfee, J. McMillan and M. D. Whinston, Multiproduct Monopoly, Commodity Bundling, and Correlation of Values, The Quartely Journal of Economics, 104 (2) (1989) 371-383.

D. L. McAdams, Multiproduct Monopoly Bundling, Mimeo, Graduate School of Business, Stanford University, Stanford, CA (1997).

F. H. Page and P. K. Monteiro, Three principles of competitive nonlinear pricing, Journal of Mathematical Economics, 39 (1-2) (2003) 63-109.

B. J. Nalebuff, Bundling, Yale ICF Working Papers Series Num. 99-14 (1999).

B. J. Nalebuff, Bundling as an Entry Barrier, The Quarterly Journal of Economics, 119 (1) (2004) 159-187.

J. C. Rochet and L. A. Stole, Nonlinear Prices with Random Participation, The Review of Economic Studies, 69 (1) (2002) 277-311.

R. Schmalensee, Gaussian Demand and Commodity Bundling, The Journal of Business, 57 (1) (1984) S211-S230.

L. S. Shapley, Complements and Substitutes in the Optimal Assignment Problem, Naval Research Logistics Quarterly, 9 (1) (1962) 45-48.

L. S. Shapley, Cores of convex games, International Journal of Game Theory, 1 (1) (1971) 11-26.

Y. Tauman, A. Urbano and J. Watanabe, A Model of Multiproduct Price Competition, Journal of Economic Theory, vol. 77 (2) (1997) 377-401.

M. D. Whinston, Tying, Foreclosure, and Exclusion, The American Economic Review, 80 (4) (1990) 837-859.