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Solidarity in games with a coalition structure*

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Abstract

A new axiomatic characterization of *the two-step Shapley value* (Kamijo (2009)) is presented based on a *solidarity* principle of the members of any union: when the game changes due to the addition or deletion of players outside the union, all members of the union will share the same gains/losses.

Keywords: Games with a coalition structure. Owen value. The two-step Shapley value. Solidarity.

1 Introduction

In the framework of cooperative games, there are many natural settings in which players organize themselves into groups for the purpose of payoff bargaining. Those include syndicates, unions, cartels, parliamentary coalitions, cities, countries, etc. This fact is incorporated into the game by a *coalition structure*, which is an exogenous partition of players into a set of groups or unions. The evaluation of players' expectations in the game is given by a *coalitional value* which takes into account the fact that agents interact on two levels: firstly, among the unions, and secondly, within each union.

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Games with a coalition structure were first considered by Aumann and Drèze (1974). They extend the value introduced in Shapley (1953) in such a way that the game splits into subgames played by the unions in isolation, and every player receives his Shapley value (Sh) in the subgame played within his union. A different approach was followed by Owen (1977). In his case, unions play a quotient game among themselves, and each union receives a payoff that is shared among its players in an internal game. Both payoffs, in the quotient game for unions and within each union for its players, are given by the Shapley value. This gives rise to the Owen value.

Alternative coalitional values have been considered too. In Owen (1982), the Banzhaf (1965) value (Bz) was used to solve both the game among unions and the game within each union. In Alonso and Fiestras (2002), the symmetric coalitional Banzhaf value was introduced, the Banzhaf value being applied in the quotient game, and the Shapley value within unions. In Amer *et al.* (2002) an example was introduced as its counterpart (reversing the application of the Shapley and Banzhaf values). These four values cover the possible variations of the application of the Shapley and Banzhaf values at the two levels of interaction: (Sh, Sh), (Bz, Bz), (Bz, Sh), and (Sh, Bz). Axiomatic characterizations of these values can be found in Alonso *et al.* (2007). These values fall into a wider family of (ψ, ϕ)-coalitional values considered in Albizuri and Zarzuelo (2004), where ψ is the semivalue applied in the game among unions, and ϕ is the semivalue applied within each union.

The standard motivation for incorporating a coalition structure into a game is that players are interested in joining a union in order to improve their bargaining position in the game. This is, for example, the point of view given in Hart and Kurz (1983) [Section 1, page 1048]:

> "With this view in mind, coalitions do not form in order to obtain their "worth" and then "leave" the game. But rather, they "stay" in the game and bargain as a unit with all the other players. This means that coalitions try to obtain as much as possible by not letting the others exploit their (individual) weaknesses when they are separated. As an everyday example of such a situation, "I will have to check this with my wife/husband" may (but not necessarily) lead to a better bargaining position, due to the fact that the other party has to convince *both* the player and the spouse."

Hence, when a union is formed, all its members commit themselves to bargaining with the others as a unit. A critical question here is how to share the gains (or losses) obtained by the players in a union. For example, suppose that in a certain marital situation, the couple follows a rule Ψ in order to fix what each one initially contributes to the union. And also suppose that if they separate in the future, Ψ will be used too to determine how to share their common future wealth. If Ψ was consistent with the marriage vows that joined them "for better or for worse", then Ψ should share their wealth variation equally between them. We can informally express this *solidarity* property as follows: if the data of the game change due to factors external to the union, then all members of the union change their value in an equal amount. Although many types of changes can be considered, this paper will only focus on addition or deletion in the set of players outside the union.

It is easy to see that the Owen value does not satisfy this solidarity property. This is because the payoff to each member is determined by the Shapley value of a new auxiliary game, played by all the members of the union (and only by them). In this auxiliary game, the worth of each subcoalition is given by its payoff (Shapley value) in a modified quotient game played by itself and the remaining unions[1]. Then if we delete players outside the union, this auxiliary game changes and the internal redistribution of the wealth obtained by the union also changes, even if the total payoff that the union obtains is unchanged.

In Kamijo (2009) a new coalitional value, named *the two-step Shapley value*, is considered and axiomatized. This value satisfies most of the properties that support the Shapley value in the setting of games without coalition structure. Therefore, it can be considered as an alternative value extension to the coalition structure setting. Our goal is the characterization of the two-step Shapley value by explicitly introducing this solidarity principle in the axiomatic system. This yields an additional support to consider the two-step Shapley value as an interesting option to replace the Owen value whenever solidarity must matter.

The rest of the paper is organized as follows. Section 2 is devoted to some preliminary definitions and notation and the two-step Shapley value is presented. Section 3 introduces

[1] It can be assumed that either the remaining players in the union leave the game, or will break apart into individuals (singletons), or into a new union. In all three cases, the payoffs obtained in this auxiliary game are the same. See Hart and Kurz (1983) for more details.

the solidarity axiom and shows that the Owen value does not satisfy this axiom. We give the axiomatic characterization of the two-step Shapley value based on this axiom. In Section 4, (i) we show that the set of axioms is independent, (ii) and we provide an alternative characterization using the axiom of balanced contributions.

2 Notation and definitions

A *cooperative game* with transferable utility (TU-game) is a pair (N, v) where N is a nonempty and finite set and $v : 2^N \to \mathbb{R}$ is a *characteristic function*, defined on the power set of N, satisfying $v(\emptyset) = 0$. An element i of N is called a *player* and every nonempty subset S of N a *coalition*. The real number $v(S)$ is called the *worth* of coalition S, and it is interpreted as the total payoff that the coalition S, if it forms, can obtain for its members. Let $\mathcal{G}^N$ denote the set of all cooperative TU-games with player set N.

Given $S \subseteq N$, we denote the restriction of (N, v) to S as (S, v). For simplicity, we write $S \cup i$ instead of $S \cup \{i\}$, $N \setminus i$ instead of $N \setminus \{i\}$, and $v(i)$ instead of $v(\{i\})$.

A *value* is a function γ which assigns to every TU-game (N, v) and every player $i \in N$, a real number $\gamma_i(N, v)$, which represents an assessment made by i of his gains from participating in the game. One of the most important values is the *Shapley value* (Shapley, 1953). The Shapley value of the game (N, v) is denoted as $Sh(N, v)$.

Let $\Omega(N)$ be the set of all orders on N. The Shapley value of a game (N, v) is given by the formula

$$Sh_i(N, v) = \frac{1}{|\Omega(N)|} \sum_{\omega \in \Omega(N)} \left[v\left(P_i^\omega(N) \cup i\right) - v\left(P_i^\omega(N)\right) \right], \quad (i \in N),$$

where $P_i^\omega(N) = \{j \in N : \omega(j) < \omega(i)\}$ and $\omega(j)$ denotes the position of j in the order ω. Thus, the Shapley value assigns to each player his expected marginal contribution with respect to a uniform distribution over all orders on N.

Two players $i, j \in N$ are *symmetric* in (N, v) if $v(S \cup i) = v(S \cup j)$ for all $S \subseteq N \setminus \{i, j\}$. Player $i \in N$ is a *dummy player* in a game (N, v) if $v(S \cup i) = v(S) + v(i)$ for all $S \subseteq N \setminus i$. Player $i \in N$ is a *null player* in (N, v) if $v(S \cup i) = v(S)$ for all $S \subseteq N \setminus i$. Given two games (N, v) and (N, w), the game $(N, v + w)$ is defined as $(v + w)(S) = v(S) + w(S)$ for all $S \subseteq N$.

Consider the following properties of a value γ in $\mathcal{G}^N$:

Efficiency: For all (N,v), $\sum_{i\in N}\gamma_i(N,v)=v(N)$.

Additivity: For all (N,v) and (N,w), $\gamma(N,v+w)=\gamma(N,v)+\gamma(N,w)$.

Symmetry: For all (N,v), if two players $i,j\in N$ are symmetric, then $\gamma_i(N,v)=\gamma_j(N,v)$.

Null player axiom: For all (N,v) and all $i\in N$, if i is a null player, then $\gamma_i(N,v)=0$.

The following theorem is due to Shapley (1953).

Theorem 1 *(Shapley, 1953) A value γ on $\mathcal{G}^N$ satisfies* efficiency, additivity, symmetry *and* null player axiom *if, and only if, γ is the Shapley value.*

Given $N\subseteq U$ finite, a *coalition structure* over N is a partition of N, i.e., $B=\{B_1,B_2,...,B_m\}\subseteq 2^N$ is a coalition structure if it satisfies $\bigcup_{1\leq k\leq m}B_k=N$ and $B_k\cap B_l=\varnothing$ when $k\neq l$. We also assume $B_k\neq\emptyset$ for all k. There are two trivial coalition structures: The first, which we denote by B^N, where only the grand coalition forms, that is, $B^N=\{N\}$; and the second is the discrete coalition structure, where each union is a singleton and is denoted by B^n, (i.e., $B^n=\{\{1\},\{2\},...,\{n\}\}$). We denote the *game* (N,v) *with coalition structure* B as (B,v). Let $\mathcal{CSG}^N$ denote the family of all TU-games with coalition structure with player set N, and let $\mathcal{CSG}$ denote the set of all TU-games with coalition structure.

Given a game $(B,v)\in\mathcal{CSG}^N$, where $B=\{B_1,B_2,...,B_m\}$, the *quotient game* is the *TU*-game $(M,v_B)\in G^M$ where $M=\{1,2,...,m\}$ and $v_B(T)=v\left(\bigcup_{i\in T}B_i\right)$ for all $T\subseteq M$. That is, (M,v_B) is the game induced by (B,v) by considering the coalitions of B as players. Notice that for the trivial coalition structure B^n we have $(M,v_{B^n})\equiv(N,v)$. We say that two unions $B_k,B_l\in B$ are *symmetric* in (B,v) if the players $k,l\in M$ are symmetric in the game (M,v_B). We say that $B_k\in B$ is a *null coalition* if player $k\in M$ is a null player in the game (M,v_B).

The evaluation of players' expectations in the game with a coalition structure is given by a *coalitional value*, which takes into account the fact that the interaction among agents is now played on two levels: firstly, among the unions as players, and secondly, among the players within each union. Formally, a *coalitional value* is a function Φ that assigns a vector in $\mathbb{R}^N$ to each game with coalition structure $(B,v)\in\mathcal{CSG}^N$. One of the most important coalitional values is the *Owen value* (Owen, 1977). We denote the Owen value of a game (B,v) as $Ow(B,v)$.

Similarly to the Shapley value, the Owen value can also be defined by orders. Let B be a coalition structure over N and $\omega \in \Omega(N)$. We say that ω is *admissible* with respect to B if for all $i, j, k \in N$, $i, k \in B_l \in B$, and $\omega(i) < \omega(j) < \omega(k)$ implies that $j \in B_l$. In other words, ω is admissible with respect to B if players of the same union in B appear successively in ω. We denote by $\Omega(B, N)$ the set of all admissible orders (on N) with respect to B. The Owen value of a game $(B, v) \in \mathcal{CSG}^N$ is given by the formula

$$Ow_i(B, v) = \frac{1}{|\Omega(B, N)|} \sum_{\omega \in \Omega(B,N)} \left[v\left(P_i^\omega(N) \cup i\right) - v\left(P_i^\omega(N)\right) \right], \quad (i \in N),$$

that is, the Owen value assigns to each player his expected marginal contribution with respect to a uniform distribution over all orders on N that are admissible with respect to the coalition structure.

For any coalitional value Φ let

$$\Phi(B, v)[S] = \sum_{i \in S} \Phi_i(B, v), \quad (S \subseteq N).$$

We present now the axioms that characterize the Owen value in $\mathcal{CSG}^N$.

(E) *Efficiency:* For all $(B, v) \in \mathcal{CSG}^N$, $\Phi(B, v)[N] = v(N)$.

(A) *Additivity:* For all (B, v) and $(B, w) \in \mathcal{CSG}^N$, $\Phi(B, v + w) = \Phi(B, v) + \Phi(B, w)$.

(NP) *Null player:* For all $(B, v) \in \mathcal{CSG}^N$ and all $i \in N$, if i is a null player in (N, v), then $\Phi_i(B, v) = 0$.

(ISy) *Intracoalitional symmetry:* For all $(B, v) \in \mathcal{CSG}^N$ and all $i, j \in N$ and all $k \in M$, if $i, j \in B_k$ $(B_k \in B)$ are symmetric in (N, v), then $\Phi_i(B, v) = \Phi_j(B, v)$.

(CSy) *Coalitional symmetry*[2]: For all $(B, v) \in \mathcal{CSG}^N$, if two unions $B_k, B_l \in B$ are symmetric, then $\Phi(B, v)[B_k] = \Phi(B, v)[B_l]$.

The following theorem is due to Owen (1977).

Theorem 2 *(Owen, 1977) A value Φ on $\mathcal{CSG}^N$ satisfies* efficiency, additivity, null player axiom, intracoalitional symmetry *and* coalitional symmetry *if, and only if, Φ is the Owen value.*

[2] Axioms *ISy* and *CSy* are often called in the literature *symmetry within unions* and *symmetry in the quotient game*, respectively.

Note that for the trivial coalition structures B^n and B^N, $Ow(B^N, v) = Ow(B^n, v) = Sh(N, v)$.

Kamijo (2009) considered the following coalitional value, named the two-step Shapley value, and defined two new axioms about null players and symmetric players in order to axiomatize it.

Definition 1 *Given a game $(B, v) \in \mathcal{CSG}^N$, the two-step Shapley value of (B, v) is given by the formula:*

$$\Psi_i(B, v) = Sh_i(B_k, v) + \frac{1}{|B_k|}\left[Sh_k(M, v_B) - v(B_k)\right], \quad (i \in B_k \in B). \tag{1}$$

The first term of Ψ is a sort of "competitive component", since it rewards each player taking into account his strategic strength in the restricted game (B_k, v), whereas the second one is common to all members of B_k and represents the "solidarity component" of the value.

Note that, since the Shapley value satisfies *efficiency*, Ψ satisfies the following relationship

$$\sum_{i \in B_k} \Psi_i(B, v) = Sh_k(M, v_B), \quad (B_k \in B).$$

We present now the axioms that characterize Ψ in $\mathcal{CSG}^N$.

(CNP) *Coalitional null player*: For all $(B, v) \in \mathcal{CSG}^N$ and all $i \in N$ and all $k \in M$, if $i \in B_k$ is a null player in (N, v), and k is a dummy player in (M, v_B), then $\Phi_i(B, v) = 0$.

(IE) *Internal Equity*: For all $(B, v) \in \mathcal{CSG}^N$ and all $i, j \in N$ and all $k \in M$, if $i, j \in B_k \in B$ are symmetric players in (B_k, v) then $\Phi_i(B, v) = \Phi_j(B, v)$.

In the statement of the *coalitional null player axiom*, the usual requirement that a null player obtains nothing in any situation is weakened so that he could obtain a positive reward because of the mutual assistance between the internal members in the coalition. *Internal equity* requires that two distinct players who are symmetric in the internal game (B_k, v) should be equally treated and thus receive the same payoff. It is clear that *intra-coalitional symmetry* is weaker than *internal equity* and that *null player axiom* is stronger than *coalitional null player axiom*.

Then the following theorem holds.

Theorem 3 (Kamijo (2009); Theorem 1) *A value Φ on $\mathcal{CSG}^N$ satisfies* efficiency, additivity, coalitional symmetry, coalitional null player axiom *and* internal equity *if, and only if,* $\Phi \equiv \Psi$.

Kamijo also showed that Ψ can be computed by means of orders. Let $(B,v) \in \mathcal{CSG}^N$ be a game. For all $\omega \in \Omega(B,N)$ and all $B_i, B_j \in B$, we write $\omega(B_i) < \omega(B_j)$ when players of B_i appear before than players of B_j at ω. For all $B_j \in B$ and all $i \in B_j$, define:

$$P_i^\omega(B_j) = P_i^\omega(N) \cap B_j; \quad T_j^\omega = \{k \in M : \omega(B_k) < \omega(B_j)\};$$

$$d_i^\omega(v) = \begin{cases} v\left(P_i^\omega(N) \cup i\right) - v\left(B_j \diagdown i\right) - v\left(\bigcup_{k \in T_j^\omega} B_k\right) & \text{if } \omega \in \Omega_i^1(B,N), \\ v\left(P_i^\omega(B_j) \cup i\right) - v\left(P_i^\omega(B_j)\right) & \text{if } \omega \in \Omega_i^2(B,N), \end{cases}$$

where

$$\begin{aligned} \Omega_i^1(B,N) &= \{\omega \in \Omega(B,N) : P_i^\omega(B_j) = B_j \diagdown i\}, \\ \Omega_i^2(B,N) &= \{\omega \in \Omega(B,N) : P_i^\omega(B_j) \subsetneq B_j \diagdown i\}. \end{aligned}$$

That is, for all orders in $\Omega_i^1(B,N)$, i is the last player completing the union B_j.

Proposition 1 (Kamijo (2009); Theorem 3) *For all $(B,v) \in \mathcal{CSG}^N$,*

$$\Psi_i(B,v) = \frac{1}{|\Omega(B,N)|} \sum_{\omega \in \Omega(B,N)} d_i^\omega(v), \qquad (i \in N).$$

3 The solidarity axiom

We now want to express the solidarity principle that guides union formation in the sense that when the game changes due to addition or deletion of players outside the union, all members of the union will share the same gains/losses. For any $h \in B_l \in B$, define $B_{-h} := (B_1, ..., B_l \backslash h, ..., B_m)$.

(PS) *Population solidarity within unions:* For all $(B,v) \in \mathcal{CSG}^N$ and all $i,j,h \in N$, where $i,j \in B_k$ and $h \in B_l$, $k \neq l$,

$$\Phi_i(B,v) - \Phi_i(B_{-h},v) = \Phi_j(B,v) - \Phi_j(B_{-h},v).$$

This axiom makes sense for coalitional values defined on $\mathcal{CSG}$, given that in this axiom the value must be applied on N and also on $N\backslash h$, for all $h \in N$.

Remark 1 *The idea that variations in population should affect all agents equally has a long standing tradition. As such, it is a strengthening of the idea of population monotonicity, introduced by Thomson (1983a, 1983b) in the context of bargaining and applied by Sprumont (1990) to standard coalitional games. A value* γ *satisfies* population monotonicity *if, for all game* $(N, v) \in G^N$ *and all* $i \in S \subseteq N$*, it holds that* $\gamma_i(N, v) \geq \gamma_i(S, v)$*. That is, if new agents join a society no initial agent should be worse off. A direct adaptation of this idea to games with a coalition structure could be expressed as follows.*

(PM) Population monotonicity within unions*: For all* $(B, v) \in \mathcal{CSG}^N$ *and all* $i, h \in N$*, where* $i \in B_k$ *and* $h \in B_l$*,* $k \neq l$*,*

$$\Phi_i(B, v) \geq \Phi_i(B_{-h}, v).$$

It must be noted that, in one hand, PM *is an* ordinal *requirement, because only forces that changes in payoffs inside the union have the same sign, whereas in* PS *the changes are of the same magnitude. On the other hand,* PM *requires that the effect of drawing a player* h *of union* B_l *must be always* negative *for players in* B_k*, but in* PS *the sign of this effect is not imposed, it is a consequence of the characteristics of the game.*

Claim 1 *The Owen value does not satisfy* population solidarity within unions.

Proof. Consider the following game[3]. The set of players is $N = \{1, 2, 3, 4\}$, and there are two commodities, say x_l is the number of "left-gloves" and x_r is the number of "right-gloves". Each player $i \in N$ has an endowment of goods, $\omega^i = (\omega_l^i, \omega_r^i)$, and the worth of each coalition $S \subseteq N$ is given by $v(S) := \min\left\{\sum_{i\in S}\omega_l^i, \sum_{i\in S}\omega_r^i\right\}$. In our example, let $\omega^1 = (1-\epsilon, 0)$, $\omega^2 = (0, 1-\epsilon)$, $\omega^3 = (\epsilon, \epsilon)$, and $\omega^4 = (\epsilon, 0)$[4]. Assume initially that only players $1, 2, 3$ are in the game, and that they act as singletons: $B_{-4}^n = \{\{1\}, \{2\}, \{3\}\}$. The Owen value is

$$Ow_1\left(B_{-4}^n, v\right) = Ow_2\left(B_{-4}^n, v\right) = \frac{1}{2} - \frac{\epsilon}{2}, \quad Ow_3\left(B_{-4}^n, v\right) = \epsilon.$$

[3] This is a variation of Shafer's Example 2, in Shafer (1980).

[4] It is assumed that ϵ is small enough. For our example, a value of ϵ less than 1/5 suffices.

If players 1 and 2 form a union $\{1,2\}$, we have the coalition structure $B = \{B_k = \{1,2\}, B_l = \{3\}\}$, but the payoffs remain unchanged:

$$Ow\left(B^n_{-4}, v\right) = Ow\left(B, v\right).$$

Suppose now that player 4 enters the game as a singleton. The resulting coalition structure is $B' = \{B_k = \{1,2\}, B_l = \{3\}, B_t = \{4\}\}$. It can be checked that B_t is a null coalition in the game (B', v). And note that $B'_{-4} = \{\{1,2\},\{3\}\} = B$. Nevertheless, easy computations yield the following payoffs:

$$\begin{aligned}
Ow_1(B', v) &= \frac{1}{2} - \frac{3}{4}\epsilon < Ow_1(B'_{-4}, v) = \frac{1}{2} - \frac{\epsilon}{2}, \\
Ow_2(B', v) &= \frac{1}{2} - \frac{1}{4}\epsilon > Ow_2(B'_{-4}, v) = \frac{1}{2} - \frac{\epsilon}{2}, \\
Ow_3(B', v) &= Ow_3(B'_{-4}, v) = \epsilon, \\
Ow_4(B', v) &= 0.
\end{aligned}$$

Alternatively, we can consider $\bar{B} = \{B_k = \{1,2\}, B_l = \{3,4\}\}$. But again $\bar{B}_{-4} = \{\{1,2\},\{3\}\} = B$ and the Owen value also yields the same payoffs, i.e., $Ow(B', v) = Ow(\bar{B}, v)$. In both cases, there is a redistribution effect in favour of player 2, although the total worth of that union $\{1,2\}$ is $1-\epsilon$, independently of whether or not player 4 is in the game. ■

Remark 2 *Casajus (2009) introduced a new coalitional value which satisfies a rather similar solidarity property called* splitting. *The main difference is that the two-step Shapley value is efficient within the grand coalition, i.e.* $\sum_{i\in N} \Psi_i(B,v) = v(N)$, *and the value introduced by Casajus satisfies efficiency within each union, i.e.* $\sum_{i\in B_k} \Psi_i(B,v) = v(B_k)$ *for all* $B_k \in B$.

We define now two additional axioms:

(NC) *Null coalition:* For all $(B,v) \in \mathcal{CSG}^N$, $\Phi\left(B,v\right)\left[B_k\right] = 0$ when $B_k \in B$ is a null coalition.

(Coh) *Coherence:* For all $(N,v) \in G^N$, $\Phi\left(B^N, v\right) = \Phi(B^n, v)$.

Coherence means that games in which all players belong to only one union and when all of them act as singletons are indistinguishable.

Remark 3 *The Owen value also satisfies* NC *and* Coh.

We wish to stress the independence between the *null player* and the *null coalition* axioms. There is no relation between *null coalition axiom* and *coalitional null player axiom* either. We illustrate these aspects with the following propositions.

Proposition 2 *The* null coalition axiom *does not imply the* null player axiom.

Proof. Consider the following coalitional value $\digamma$ defined by

$$\digamma_i (B, v) = \frac{Sh_k(M, v_B)}{|B_k|}, \forall i \in B_k, \forall B_k \in B.$$

$\digamma$ satisfies *null coalition axiom* since the Shapley value satisfies *null player axiom*. Let (B, v) be the game defined by $N = \{1, 2, 3\}$, $B = \{B_k, B_l\}$, with $B_k = \{1, 2\}$ and $B_l = \{3\}$, and v is given by $v(1) = 0$, $v(2) = 1$, $v(3) = 2$, $v(1, 2) = 1$, $v(1, 3) = 2$, $v(2, 3) = 4$ and $v(N) = 4$. Player 1 is a null player and nevertheless

$$\digamma_1 (B, v) = \digamma_2 (B, v) = 3/4, \quad \digamma_3 (B, v) = 5/2.$$

■

Note that, although *NC* does not imply *NP*, it is true that *null coalition axiom* implies the following weaker version of the *null player axiom*:

(NP*) *Null player in singletons:* For all $(B, v) \in \mathcal{CSG}^N$ and all $i \in N$, if i is a null player in (N, v) and $\{i\} \in B$, then $\Phi_i (B, v) = 0$.

Proposition 3 *The* null player axiom *does not imply the* null coalition axiom.

Proof. Define the following coalitional value:

$$\Gamma (B, v) = Sh(N, v), \ \forall (B, v) \in \mathcal{CSG}^N$$

Taking into account the properties of the Shapley value, Γ satisfies *null player axiom*. Let (B, v) be the game defined by $N = \{1, 2, 3\}$, $B = \{B_k, B_l\}$, with $B_k = \{1, 2\}$ and $B_l = \{3\}$, and v is given by $v(1) = v(2) = 0$, $v(3) = 3$, $v(1, 2) = 0$, $v(1, 3) = 4$, $v(2, 3) = 3$ and $v(N) = 3$. Union $\{1, 2\}$ is a null coalition, but $\Gamma (B, v) = (1/6, -2/6, 19/6)$, then $\Gamma_1 (B, v) + \Gamma_2 (B, v) = -1/6 \neq 0$, so Γ does not satisfy *null coalition axiom.* ■

Proposition 4 *The* null coalition axiom *does not imply the* coalitional null player axiom.

Proof. The coalitional value $\digamma$ defined in Proposition 2 does not satisfy *coalitional null player axiom* because N is always a dummy coalition in $\left(B^N, v\right)$ and $\digamma_i\left(B^N, v\right) = v(N)/\left|N\right|, \forall i \in N$. So, with the same v as in Proposition 2, player 1 is a null player and $\digamma_1\left(B^N, v\right) \neq 0$. ∎

Corollary 1 Coalitional null player axiom *does not imply* null coalition axiom.

Proof. If *coalitional null player axiom* implied *null coalition axiom*, since *NP* implies *CNP*, then *null player axiom* would imply *null coalition axiom.* ∎

We are now ready to present the new axiomatic characterization of the value Ψ.

Theorem 4 *A value* Φ *on* $\mathcal{CSG}$ *satisfies* efficiency, additivity, coalitional symmetry, null coalition axiom, coherence *and* population solidarity within unions *if, and only if,* $\Phi \equiv \Psi$.

Proof. *Existence.* Let $(B, v) \in \mathcal{CSG}^N$. Since the Shapley value satisfies *efficiency*, for each $B_k \in B$ we have that $\sum_{i \in B_k} \Psi_i(B, v) = Sh_k(M, v_B)$, and then $\sum_{i \in N} \Psi_i(B, v) = \sum_{k \in M} \sum_{i \in B_k} \Psi_i(B, v) = \sum_{k \in M} Sh_k(M, v_B) = v(N)$. Thus Ψ satisfies *efficiency.* Moreover, since the Shapley value satisfies *additivity*, Ψ also satisfies *additivity.*

It is straightforward that Ψ satisfies *coalitional symmetry* because the Shapley value satisfies *symmetry* and $\sum_{i \in B_k} \Psi_i(B, v) = Sh_k(M, v_B), \forall k \in M$.

In order to prove that Ψ satisfies *null coalition axiom*, note that if $B_k \in B$ is a null coalition, then $k \in M$ is a null player in (M, v_B), hence, as the Shapley value satisfies *null player axiom*, then $Sh_k(M, v_B) = 0$ and therefore $\sum_{i \in B_k} \Psi_i(B, v) = Sh_k(M, v_B) = 0$.

In order to prove *coherence*, if $B = B^N$, then $M = \{1\}$, $Sh_1(M, v_B) = v(N)$, and then $\Psi_i\left(B^N, v\right) = Sh_i(N, v) + \frac{1}{|N|}\left[v(N) - v(N)\right] = Sh_i(N, v)$ for all $i \in N$. On the other hand, if $B = B^n$ then $(M, v_B) = (N, v)$, so $\Psi_i(B^n, v) = v(i) + Sh_i(N, v) - v(i) = Sh_i(N, v)$. Therefore $\Psi\left(B^N, v\right) = \Psi(B^n, v) = Sh(N, v)$.

Taking into account the definition of Ψ, for any $h \in B_l \in B$ and all $i, j \in B_k$, $l \neq k$:

$$\begin{aligned} \Psi_i(B, v) - \Psi_i(B_{-h}, v) &= \frac{1}{|B_k|}\left[Sh_k(M, v_B) - Sh_k(M', v_{B_{-h}})\right] = \\ &= \Psi_j(B, v) - \Psi_j(B_{-h}, v), \end{aligned}$$

where $M' = M \backslash l$ if $|B_l| = 1$, $M = M'$ otherwise. So Ψ satisfies *population solidarity within unions.*

Uniqueness. Let Φ be a coalitional value satisfying the above axioms and let $(B,v) \in \mathcal{CSG}^N$ be a game. We define the following value γ on $\mathcal{G}^M$ by:

$$\gamma_k(M, v_B) = \Phi(B,v)[B_k], \quad (k \in M).$$

In order to see that γ is well-defined, suppose there exist two games u and v such that $u_B = v_B$. We prove that $\Phi(B,u)[B_k] = \Phi(B,v)[B_k]$, for all $k \in M$. Indeed, since $u_B = v_B$, all the unions $B_k \in B$ are null in $(B, N, u-v)$[5]. Then, by the *null coalition axiom*, $\Phi(B, u-v)[B_k] = 0$, for all $B_k \in B$, and by *additivity*,[6]

$$\begin{aligned} 0 &= \Phi(B,u-v)[B_k] = \Phi(B,u)[B_k] + \Phi(B,-v)[B_k] \\ &= \Phi(B,u)[B_k] - \Phi(B,v)[B_k], \quad (k \in M). \end{aligned}$$

Now, since Φ satisfies *efficiency*, *additivity*, *coalitional symmetry* and *null coalition axiom*, then by Theorem 1, $\gamma_k(M,v_B) = \Phi(B,v)[B_k] = Sh_k(M,v_B)$ for all $B_k \in B$. In particular, this expression jointly with *coherence* imply that $\Phi(B^n,v) = Sh(N,v) = \Phi\left(B^N,v\right)$. Thus Φ is uniquely determined for the two trivial coalition structures.

Suppose now that $|B| \geq 2$ and let $B_k \in B$. By *population solidarity within unions*,

$$\Phi_i(B,v) - \Phi_i(B_{-h},v) = d_k, \text{ for all } i \in B_k \text{ and all } h \in B_l \neq B_k,$$

and then

$$\Phi(B,v)[B_k] - \Phi(B_{-h},v)[B_k] = Sh_k(M,v_B) - Sh_k(M',v_{B_{-h}}) = |B_k|\, d_k,$$

hence

$$\Phi_i(B,v) = \Phi_i(B_{-h},v) + \frac{1}{|B_k|}\left[Sh_k(M,v_B) - Sh_k(M',v_{B_{-h}})\right], \quad (i \in B_k).$$

Applying *population solidarity within unions* repeatedly so that all coalitions except B_k leave the game, we finally obtain

$$\begin{aligned} \Phi_i(B,v) &= \Phi_i(B^{B_k},v) + \frac{1}{|B_k|}\left[Sh_k(M,v_B) - Sh_k(\{k\},v_{\{B_k\}})\right] = \\ &= Sh_i(B_k,v) + \frac{1}{|B_k|}\left[Sh_k(M,v_B) - v(B_k)\right] = \Psi_i(B,v), \quad (i \in B_k). \end{aligned}$$

■

[5] The game $u-v$ is defined as $(u-v)(S) = u(S) - v(S)$, for all $S \subset N$.

[6] In particular, $0 = \Phi(B,v-v)[B_k] = \Phi(B,v)[B_k] + \Phi(B,-v)[B_k]$, then $\Phi(B,-v)[B_k] = -\Phi(B,v)[B_k]$.

Remark 4 *The coalitional value* Ψ *also satisfies* intracoalitional symmetry*. This follows from the fact that if* $i, j \in B_k \in B$ *are symmetric in* (N, v) *they are also symmetric in* (B_k, v)*. By the* symmetry *of the Shapley value and the definition of* Ψ *it follows that* $\Psi_i(B, v) = \Psi_j(B, v)$*.*

An important fact which distinguishes the coalitional value Ψ from the Owen value is the *null player axiom.* The Owen value satisfies *null player axiom* and Ψ does not. If $i \in B_k$ is a null player in (N, v), he obtains $\frac{1}{|B_k|}[Sh_k(M, v_B) - v(B_k)]$ which in general is different from zero. This fact is in accordance with the solidarity principle: as far as a null player is accepted in the union, he gets the same benefits as any other member in the union (all the members "are on the same boat")[7]. A different question altogether arises when the coalition structure is not given a priori, and we want to make a stability analysis of which coalition structure will be finally formed (as done in Hart and Kurz, 1983). In that case, if we use the coalitional value Ψ, a union B_k for which $Sh_k(M, v_B) - v(B_k) > 0$ will not be stable if some null player h belongs to B_k, because by excluding h from the union and forming the new coalition structure $B' = \left\{(B_l)_{l \neq k}, B_k \backslash h, \{h\}\right\}$, all the members in $B_k \backslash h$ increase their payoffs: as h is a null player in (N, v), then $v(B_k) = v(B_k \backslash h)$, and in the new quotient game $(M', v_{B'})$, where $M' = M \cup \{h\}$, we have that $Sh_k(M', v_{B'}) = Sh_k(M, v_B)$, and then

$$\Psi_i(B', v) - \Psi_i(B, v) = \frac{1}{|B_k| \cdot |B_k \backslash h|}[Sh_k(M, v_B) - v(B_k)], \quad (i \in B_k \backslash h).$$

We summarize this section with a table of properties that Ψ and the Owen value satisfy (* means that the property is used in the characterization of the value).

	E	A	CSy	Coh	NC	CNP	ISy	IE	PS	NP
Ψ(on $\mathcal{CSG}$)	**yes***	**yes***	**yes***	**yes***	**yes***	*yes*	*yes*	*yes*	**yes***	*no*
Ψ (Kamijo)	**yes***	**yes***	**yes***	*yes*	*yes*	**yes***	*yes*	**yes***	*yes*	*no*
Ow	**yes***	**yes***	**yes***	*yes*	*yes*	*yes*	**yes***	*no*	*no*	**yes***

Remark 5 *Note that the characterizations of Owen (1977) and Kamijo (2009) hold true on* $\mathcal{CSG}^N$*, for all* $N \subseteq U$ *and, therefore, they also hold when viewing the two coalitional*

[7]Returning to the marriage example, if the couple has a child, he can be considered as a null member of the family during his childhood.

values as values on $\mathcal{CSG}$. Nevertheless, our characterization of Ψ by means of the population solidarity axiom only holds on $\mathcal{CSG}$, because in this axiom the value must be applied on N and $N\backslash h$.

4 Complementary results

4.1 Independence of the axiomatic system

The axiom system is independent. Indeed:

(i) *Population solidarity within unions:* The Owen value satisfies all axioms, except *population solidarity within unions*. See Claim 1 of Section 3.

(ii) *Coherence:* Let the coalitional value $\digamma^1$ be defined as

$$\digamma_i^1(B,v) = \frac{Sh_k(M, v_B)}{|B_k|}, \quad (i \in B_k \in B).$$

$\digamma^1$ satisfies all the axioms except *coherence*, because $\digamma_i^1\left(B^N, v\right) = v(N)/\left|N\right|$ and $\digamma_i^1(B^n, v) = Sh_i(N, v)$, for all $i \in N$.

(iii) *Null coalition axiom:* Let $\digamma^2$ be defined as

$$\digamma_i^2(B,v) = \frac{v(N)}{|B_k|\,|M|}, \quad (i \in B_k \in B).$$

$\digamma^2$ satisfies all the axioms except *null coalition axiom*.

(iv) *Coalitional symmetry:* Define the following coalitional value $\digamma^3$ by

a) If $N = \{1,2,3\}$ and $B = \{\{1,2\},\{3\}\}$, then:

$$\begin{cases} \digamma_1^3(B,v) = \frac{v(\{1,2\})+v(1)-v(2)}{2} \\ \digamma_2^3(B,v) = \frac{v(\{1,2\})+v(2)-v(1)}{2} \\ \digamma_3^3(B,v) = v(N) - v(\{1,2\}) \end{cases}$$

b) Otherwise, $\digamma_i^3(B,v) = \Psi_i(B,v)$, for all $i \in N$.

It is easy to see that $\digamma^3$ satisfies all the axioms except *coalitional symmetry*, because in a), $\digamma_1^3(B,v) + \digamma_2^3(B,v) = v(\{1,2\})$ is, in general, different from $\digamma_3^3(B,v) = v(N) - v(\{1,2\})$, although coalitions $\{1,2\}$ and $\{3\}$ are symmetric.

(v) *Efficiency:* Let π be any semivalue other than the Shapley value (see Dubey et al. (1981)). We define the following coalitional value Φ^π by

a) If B is different from B^n or B^N, then:

$$\Phi_i^{\pi}\left(B,v\right)=Sh_i\left(B_k,v\right)+\frac{1}{|B_k|}\left[\pi_k\left(M,v_B\right)-v\left(B_k\right)\right],\quad \left(i\in B_k\in B\right).$$

b) $\Phi^{\pi}\left(B^n,v\right)=\Phi^{\pi}(B^N,v)=Sh(N,v)$.

The coalitional value Φ^{π} satisfies all the axioms except *efficiency.* When π is the Shapley value, we obtain our coalitional value Ψ which is the only one that satisfies *efficiency.*

(vi) *Additivity:* Define the coalitional value F^4 by

a) If $(B,v)\in\mathcal{CSG}^N$ verifies that $B\neq B^n,B^N$, and $\sum\limits_{k\in M}v\left(B_k\right)\neq 0$, then:

$$F_i^4\left(B,v\right)=Sh_i\left(B_k,v\right)+\frac{1}{|B_k|}\left[\frac{v\left(B_k\right)}{\sum\limits_{k\in M}v\left(B_k\right)}v(N)-v\left(B_k\right)\right],\quad \left(i\in B_k\in B\right).$$

b) Otherwise $F^4\left(B,v\right)=\Psi(B,v)$.

The coalitional value F^4 satisfies all the axioms except *additivity.*

4.2 Properties of Balanced Contributions

This section provides another characterization for the coalitional value Ψ based on the principle of balanced contributions.

Myerson (1980) introduced this principle to characterize the Shapley value jointly with efficiency. Consider the following property of a value γ on $\mathcal{G}^N$:

(BC) *Balanced Contributions:* For all (N,v) and all $i,j\in N$,

$$\gamma_i(N,v)-\gamma_i(N\diagdown j,v)=\gamma_j(N,v)-\gamma_j(N\diagdown i,v).$$

This property states that for any two players, the amount that each player would gain or lose by the other player's withdrawal from the game should be equal.

Calvo et al. (1996) used the same principle to axiomatize the *level structure value.* This value was considered in Winter (1989) and is an extension of the Owen value for several levels of cooperation (union of players, union of union of players, and so on). In the particular case of games with a coalition structure (a single level), Calvo et al. (1996) proved that the Owen value is the only efficient coalitional value that satisfies the two following properties:

(IBC) *Intracoalitional Balanced Contributions:* For all (B,v) and all $i,j \in B_k \in B$,

$$\Phi_i(B,v) - \Phi_i(B_{-j},v) = \Phi_j(B,v) - \Phi_j(B_{-i},v).$$

(CBC) *Coalitional Balanced Contributions:* For all (B,v) and all B_i, $B_j \in B$,

$$\Phi(B,v)\left[B_i\right] - \Phi(B \diagdown B_j, v)\left[B_i\right] = \Phi(B,v)\left[B_j\right] - \Phi(B \diagdown B_i, v)\left[B_j\right].$$

In the *IBC* property, the principle of balanced contributions is applied inside a union. The *CBC* property states that for any two coalitions $B_i, B_j \in B$, the contribution of B_i to the total payoff of the members in B_j must be equal to the contribution of B_j to the total payoff of the members in B_i, hence balanced contributions is applied between unions.

We now show that the coalitional value Ψ can also be characterized with the *CBC* property.

Theorem 5 *The coalitional value Ψ is the only one that satisfies* efficiency, coalitional balanced contributions, population solidarity within unions *and* coherence*.*

Proof. To prove *existence*, we only need to show that Ψ satisfies *CBC*. Given (B,v), for all $B_k, B_l \in B$, we have that $\Psi(B,v)\left[B_k\right] = Sh_k\left(M, v_B\right)$, and $\Psi(B,v)\left[B_l\right] = Sh_l\left(M, v_B\right)$, then Ψ satisfies *CBC* if and only if

$$Sh_k\left(M, v_B\right) - Sh_k\left(M \diagdown l, v_B\right) = Sh_l\left(M, v_B\right) - Sh_l\left(M \diagdown k, v_B\right).$$

And this is true because the Shapley value satisfies *BC*.

In order to prove *uniqueness*, let Φ be a coalitional value satisfying the above axioms. Given a game (N,v) and applying *CBC* for $B = B^n$ we have:

$$\Phi_i(B^n,v) - \Phi_i(B^n_{-j},v) = \Phi_j(B^n,v) - \Phi_j(B^n_{-i},v),$$

for all $i,j \in N$. And due to the characterization of Myerson (1980), this expression jointly with *efficiency* imply that $\Phi(B^n,v) = Sh(N,v)$ for all game (N,v). By *coherence*, we have that $\Phi(B^N,v) = \Phi(B^n,v) = Sh(N,v)$ for all game (N,v). Thus Φ is uniquely determined when $|B| = 1$.

We now use induction on $|B|$. Let us assume that the uniqueness is established for $|B| \leq k$ and let (B,v) be a game such that $|B| = k+1$. By *CBC,* for all $B_i, B_j \in B$:

$$\Phi(B,v)\left[B_i\right] - \Phi(B,v)\left[B_j\right] = \Phi(B \diagdown B_j, v)\left[B_i\right] - \Phi(B \diagdown B_i, v)\left[B_j\right]. \tag{2}$$

The induction hypothesis yields

$$\begin{cases} \Phi(B\diagdown B_j, v)\,[B_i] = \Psi(B\diagdown B_j, v)\,[B_i] \\ \Phi(B\diagdown B_i, v)\,[B_j] = \Psi(B\diagdown B_i, v)\,[B_j] \end{cases}$$

And, because Ψ satisfies *CBC*, it holds that

$$\Psi(B\diagdown B_j, v)\,[B_i] - \Psi(B\diagdown B_i, v)\,[B_j] = \Psi(B, v)\,[B_i] - \Psi(B, v)\,[B_j]\,.$$

Therefore, using (2):

$$\begin{aligned} \Phi(B, v)\,[B_i] - \Phi(B, v)\,[B_j] &= \Psi(B, v)\,[B_i] - \Psi(B, v)\,[B_j] \Rightarrow \\ \Phi(B, v)\,[B_i] - \Psi(B, v)\,[B_i] &= \Phi(B, v)\,[B_j] - \Psi(B, v)\,[B_j]\,, \end{aligned}$$

for all $B_i, B_j \in B$. And then, by *efficiency*,

$$\Phi(B, v)\,[B_l] = \Psi(B, v)\,[B_l]\,, \text{ for all } B_l \in B. \tag{3}$$

We now prove that $\Phi(B, v) = \Psi(B, v)$. Let $B_l \in B$. If $|B_l| = 1$, expression (3) means that $\Phi_j(B, v) = \Psi_j(B, v)$ for $\{j\} = B_l$. Suppose that $|B_l| \geq 2$. By *population solidarity within unions* we have, for all $i, j \in B_l$ and all $B_k \neq B_l$:

$$\Phi_i\,(B, v) - \Phi_i\,(B \setminus B_k, v) = \Phi_j\,(B, v) - \Phi_j\,(B \setminus B_k, v)\,. \tag{4}$$

By the induction assumption:

$$\begin{cases} \Phi_i\,(B \setminus B_k, v) = \Psi_i\,(B \setminus B_k, v) \\ \Phi_j\,(B \setminus B_k, v) = \Psi_j\,(B \setminus B_k, v) \end{cases}$$

Hence, using (4):

$$\begin{aligned} \Phi_i\,(B, v) - \Phi_j\,(B, v) &= \Psi_i\,(B, v) - \Psi_j\,(B, v) \Rightarrow \\ \Phi_i\,(B, v) - \Psi_i\,(B, v) &= \Phi_j\,(B, v) - \Psi_j\,(B, v)\,, \quad (i, j \in B_l)\,. \end{aligned}$$

And taking (3) into account, we conclude that $\Phi_i\,(B, v) = \Psi_i\,(B, v)$, for all $i \in B_l$. ■

Remark 6 *The advantage of the characterization given in Theorem 5 is that, since the additivity property is not used, it can be applied to any subdomain of games with a coalition structure without violating uniqueness. A paradigmatic example is the case of* simple games, *a subdomain that is not closed under addition of games, with an extensive application to political sciences. The two-step Shapley value seems an interesting alternative to the Owen value for the computation of the power that political parties have in parliaments under different coalition configurations.*

In the proof of Theorem 5 *coherence* is necessary only to prove that if $B = B^N$ the solution coincides with the Shapley value. But this is induced by *IBC* and *efficiency*. Therefore, the difference between the Owen value and Ψ is based on the difference between *IBC* and *population solidarity within unions*.

Kamijo (2006, 2007) considered the following variation of the *coalitional balanced contributions*:

(GBC) *Group Balanced Contributions*[8]*:* For all (B, v) with $|B| \geq 2$, for all $i \in B_k \in B$ and for all $j \in B_h \in B$, $B_k \neq B_h$,

$$\Phi_i(B, v) - \Phi_i(B \diagdown B_h, v) = \Phi_j(B, v) - \Phi_j(B \diagdown B_k, v).$$

In this axiom, two players in distinct unions are equally affected by the deletion of the union associated with the other player. With the properties of *efficiency*, *coherence* and *group balanced contributions* he characterized the *collective value*. The collective value, Ω, is a weighted version of Ψ, where the weights are proportional to the sizes of the unions which define the coalition structure. Formally:

$$\Omega_i\left(B, v\right) = Sh_i\left(B_k, v\right) + \frac{1}{|B_k|}\left[Sh_k^w(M, v_B) - v\left(B_k\right)\right], \quad \left(i \in B_k \in B\right),$$

where Sh^w is the weighted Shapley value (Kalai and Samet, 1987), with weights w_k proportional to $|B_k|$, for all $B_k \in B$[9].

In the proof of Theorem 5 *population solidarity within unions* is used only when a whole union is deleted (see expression (4) in the proof), but this is induced by *GBC*. Thus, the difference between the collective value and Ψ is based on the difference between *GBC* and *CBC*. In fact, *CBC* and *population solidarity within unions* could be replaced in Theorem 5[10] by the following property:

(ABC) *Aggregate Balanced Contributions:* For all (B, v) with $|B| \geq 2$, for all $i \in B_k \in B$ and for all $j \in B_h \in B$, $B_k \neq B_h$,

$$|B_k|\left[\Phi_i(B, v) - \Phi_i(B \diagdown B_h, v)\right] = |B_h|\left[\Phi_j(B, v) - \Phi_j(B \diagdown B_k, v)\right].$$

[8]In Kamijo (2007) this property changes its name to *collective balanced contributions.*

[9]Two weighted versions of the Owen coalitional value can be found in Levy and McLean (1989) and Vidal-Puga (2006).

[10]The proof is left to the reader.

We summarize this section with a table of properties used in the characterization of these values.

	E	IBC	CBC	Coh	PS	ABC	GBC
Ow	**yes***	**yes***	**yes***	*yes*	*no*	*no*	*no*
Ψ	**yes***	*no*	**yes***	**yes***	**yes***	*yes*	*no*
Ψ	**yes***	*no*	*yes*	**yes***	*yes*	**yes***	*no*
Ω	**yes***	*no*	*no*	**yes***	*yes*	*no*	**yes***

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